

Second order hydrodynamic coefficients from 3-point stress tensor correlators via AdS/CFT

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Abstract

We study second order relativistic viscous hydrodynamics in 4-dimensional conformal field theories. We derive Kubo-type relations for second order hydrodynamic coefficients in terms of 3-point stress tensor retarded correlators. For $\mathcal{N}=4$ super Yang-Mills theory at strong coupling and at finite temperature we compute these stress tensor 3-point correlators, using AdS/CFT, by evaluating *real-time* cubic Witten diagrams in the AdS-Schwarzschild background. The small momentum expansion of the 3-point correlators in terms of first and second order hydrodynamic coefficients is matched with the AdS result. We arrive at the same expressions for the hydrodynamic coefficients which multiply terms quadratic in the shear and vorticity tensors in the hydrodynamic expansion of the stress tensor as did Bhattacharyya, Hubeny, Minwalla and Rangamani [1]. Our method extends the results of Baier et al [2], and allows for a unified treatment of hydrodynamic coefficients, which are extracted from 2-, and now, 3-point retarded stress tensor correlators in the AdS-Schwarzschild background.

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1 Introduction and Summary

One of the most familiar applications of AdS/CFT [3], [4] is computing linearized hydrodynamics for a variety of gauge theories with holographic duals (see [5] and references therein). More recently, second order hydrodynamic coefficients have been extracted using two different methods. Baier et al [2] used Kubo-like formulae for 2-point stress tensor correlators to access a certain subset of second-order hydro coefficients of $\mathcal{N}=4$ super Yang-Mills plasma. Their computation was made possible by a real-time finite temperature prescription for computing 2-point correlators [6, 7]. On the other hand, Bhattacharya et al [1] wrote a generalized black brane metric ansatz, dependent on the temperature and the black branes velocity viewed as collective fields, and turned the Einstein equations of motion solved perturbatively in the collective modes near the AdS boundary into equations of fluid dynamics. Specifically, the gravitational stress tensor, expanded near the boundary of AdS, took the form of a non-linear fluid dynamics stress tensor, with the various terms in the expansion in velocity and gradients multiplied by the corresponding hydrodynamic coefficients. The subsets of hydro coefficients computed by [2] and [1] agreed where they overlapped, and between them determined the full set of coefficients. By now, the second order hydrodynamic coefficients have been computed following [1] in a variety of cases: at finite chemical potential [8, 9], or in the presence of fundamental matter [10]. For recent review papers see [11] and [12]. The breakdown of second order hydrodynamics is investigated in [13].

Our work is intended as a continuation of [2], where Kubo-like formulae are used in conjunction with higher-order stress tensor correlators to extract the second order hydro coefficients computed by [1]. We first derive the necessary Kubo relations using the method of Moore and Sohrabi [14]¹, then compute 3-point retarded stress tensor correlators in the hydrodynamic regime (i.e. in the limit of small momenta) in the AdS-Schwarzschild background which is dual to finite-temperature strongly coupled $\mathcal{N}=4$ super Yang-Mills in the limit of large number of colors $N_c \gg 1$. In other words, as opposed to [1], we do not deform the gravitational background, but instead compute higher-order correlators in the background. The problem of computing such higher-order real-time finite temperature correlators was solved in [15]. For momentum-space retarded 3-point stress tensor correlators this amounts to computing real-time Witten diagrams, depicted in Fig. 1b, with three causal (two advanced and one retarded) graviton bulk-to-boundary propagators, joined at a bulk vertex which is integrated up to the black hole horizon (i.e. in the maximal causal diamond)².

To set up the problem, let us begin by revisiting some of the definitions and results of [2, 14].

The stress tensor of a conformal fluid can be written in terms of an equilibrium piece plus an extra term, which in the hydrodynamic regime can be series-expanded in gradients:

$$T^{\mu\nu} = T_{\text{eq}}^{\mu\nu} + \Pi^{\mu\nu}, \quad T_{\text{eq}}^{\mu\nu} = (\epsilon + P)U^\mu U^\nu + P g^{\mu\nu} \quad (1)$$

$$\begin{aligned} \Pi^{\mu\nu} = & -\eta\sigma^{\mu\nu} + \eta\tau_\Pi \left(\langle U \cdot \nabla \sigma^{\mu\nu} \rangle + \frac{1}{3}(\nabla \cdot U)\sigma^{\mu\nu} \right) + \kappa \left(R^{\langle\mu\nu\rangle} - 2U_\rho U_\sigma R^{\rho\langle\mu\nu\rangle\sigma} \right) \\ & + \lambda_1 \sigma^{\langle\mu}{}_\rho \sigma^{\nu\rangle\rho} + \lambda_2 \sigma^{\langle\mu}{}_\rho \Omega^{\nu\rangle\rho} + \lambda_3 \Omega^{\langle\mu}{}_\rho \Omega^{\nu\rangle\rho} + \dots \end{aligned} \quad (2)$$

¹Readers should be warned that there is an error in the derivation of the Kubo relations for λ_1 , λ_2 and λ_3 in [14]. Specifically, they left out the $(\epsilon + P)U^x U^x + P g^{xx}$ term in deriving their T^{xx} . Also, $U^i = O(h^2)$ only in the static limit.

²See also [16] for similar causality considerations and [17] for a different take on this subject.

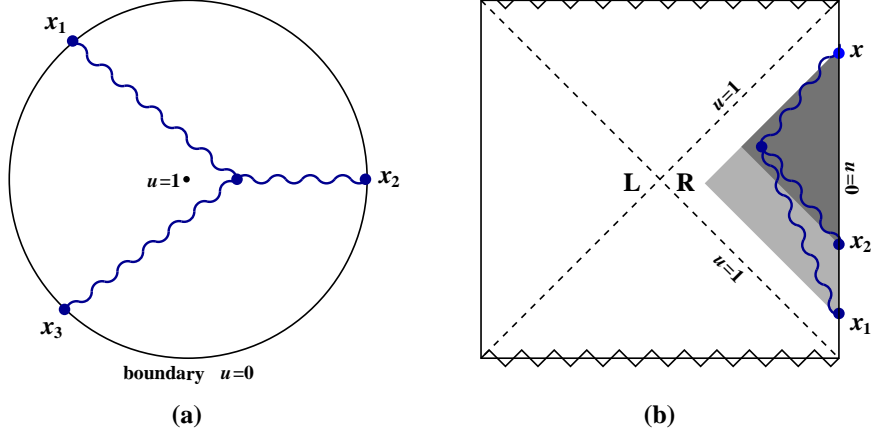


Figure 1: Witten diagram for (a) 3-point correlator in imaginary time AdS-Schwarzschild and (b) retarded 3-point correlator of type raa with the boundary point x having the largest time; x_1 and x_2 can have any time order.

Here U^μ is the fluid's velocity, normalized such that

$$U^\mu U^\nu g_{\mu\nu} = -1, \quad (3)$$

and $\Pi^{\mu\nu}$ is defined such that $U_\mu \Pi^{\mu\nu} = 0$. $g_{\mu\nu}$ is the background metric, whose curvature tensor is $R_{\rho\mu\nu}{}^\sigma$, and all derivatives are covariant with respect to the background metric. $\sigma^{\mu\nu}$, $\Omega^{\mu\nu}$ are the shear and vorticity tensors respectively:

$$\sigma^{\mu\nu} = 2\nabla^{(\mu} U^{\nu)} \equiv \Delta^{\mu\rho} \Delta^{\nu\sigma} (\nabla_\rho U_\sigma + \nabla_\sigma U_\rho) - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} \nabla_\rho U_\sigma \quad (4)$$

$$\Omega^{\mu\nu} = \frac{1}{2} \Delta^{\mu\rho} \Delta^{\nu\sigma} (\nabla_\rho U_\sigma - \nabla_\sigma U_\rho) \quad (5)$$

where $\Delta^{\mu\nu}$ are transverse (to the fluid's velocity) projectors:

$$\Delta^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu. \quad (6)$$

Angular brackets denote transverse projection, followed by symmetrization and removal of a trace:

$$O^{(\mu\nu)} = \frac{1}{2} \Delta^{\mu\rho} \Delta^{\nu\sigma} (O_{\rho\sigma} + O_{\sigma\rho}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} O_{\rho\sigma} \quad (7)$$

For a conformal fluid, energy density and pressure are related by the condition that the stress tensor is traceless, and so $\epsilon = (d-1)P$, where d is the number of space-time dimensions.

In [14], the fluid's response to a small metric perturbation was computed by solving the stress tensor conservation law

$$\nabla_\mu T^{\mu\nu} = 0, \quad (8)$$

together with the condition that the fluid describes a conformal theory,

$$T^\mu{}_\mu = 0, \quad (9)$$

order-by-order in a double series expansion in the metric fluctuation and in gradients. This expansion of the stress tensor is then compared with

$$\begin{aligned}\langle T^{\mu\nu}(\mathbf{z}) \rangle_h &= \langle T^{\mu\nu} \rangle_{h=0} - \frac{1}{2} \int d^4\mathbf{x} G_{\text{ra}}^{\mu\nu|\rho\sigma}(\mathbf{z}; \mathbf{x}) h_{\rho\sigma}(\mathbf{x}) \\ &+ \frac{1}{8} \int d^4\mathbf{x} \int d^4\mathbf{y} G_{\text{raa}}^{\mu\nu|\rho\sigma|\tau\zeta}(\mathbf{z}; \mathbf{x}, \mathbf{y}) h_{\rho\sigma}(\mathbf{x}) h_{\tau\zeta}(\mathbf{y}) + \dots\end{aligned}\quad (10)$$

where $G_{\text{ra}\dots\text{a}}^{\mu\nu|\dots}$ are retarded n -point correlators, with the measurement point \mathbf{z} having the largest time. We assume the metric fluctuations $h_{\rho\sigma}$ vanish in the far past. To avoid clutter, for the rest of the paper we will suppress the ra, raa subscripts signifying retarded correlators³ with the understanding that all the correlators we compute are of this type.

By identifying the two series expansions, namely the solution to (8,9) and the expansion in retarded correlators (10), one gains access to the hydrodynamic expansion coefficients in terms of causal stress tensor correlators. For example, the shear viscosity, η , can be computed from 2-point correlators expanded to linear order in gradients [19]. Similarly, $\eta\tau_\Pi$ and κ can also be computed from 2-point correlators, expanded to second order in gradients [2]. λ_1 , λ_2 and λ_3 require 3-point correlators, expanded to second order in gradients [14].

Since solving (8) is easier in momentum space, we will write the solution to (8) and (9) as a series expansion in momenta. For a direct comparison with the response of the stress tensor we need then the Fourier transform of (10):

$$\begin{aligned}\langle T^{\mu\nu}(\mathbf{q}) \rangle_h &= \langle T^{\mu\nu} \rangle_{h=0} - \frac{1}{2} \int d^4\mathbf{q}_1 \frac{1}{(2\pi)^4} \delta^4(\mathbf{q} - \mathbf{q}_1) G^{\mu\nu|\rho\sigma}(\mathbf{q}; -\mathbf{q}_1) h_{\rho\sigma}(\mathbf{q}_1) \\ &+ \frac{1}{8} \int d^4\mathbf{q}_1 \int d^4\mathbf{q}_2 \frac{1}{(2\pi)^4} \delta^4(\mathbf{q} - \mathbf{q}_1 - \mathbf{q}_2) G^{\mu\nu|\rho\sigma|\tau\zeta}(\mathbf{q}; -\mathbf{q}_1, -\mathbf{q}_2) h_{\rho\sigma}(\mathbf{q}_1) h_{\tau\zeta}(\mathbf{q}_2) + \dots\end{aligned}\quad (11)$$

where we have used translation invariance to factor out momentum conservation delta-functions. In a 3-point function, there are two independent momenta, and the spatial momenta could point in different directions. However, for simplicity, we will assume that they *both* point in the z direction (i.e. we consider metric fluctuations which are independent of the x, y coordinates):

$$\mathbf{q}_1^\mu = (\omega_1, 0, 0, k_1), \mathbf{q}_2^\mu = (\omega_2, 0, 0, k_2). \quad (12)$$

The purpose of this paper is to obtain the remaining second order hydrodynamic coefficients λ_1 , λ_2 and λ_3 from 3-point stress tensor correlators. We will find it simplest to study the xy -component $\langle T^{xy} \rangle_h$ and will extract second-order hydrodynamic coefficients from $G^{xy|\dots| \dots}$:

$$\begin{aligned}\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G^{xy|xz|yz} &= -\lambda_1 + \eta\tau_\Pi \\ \lim_{\substack{\omega_1 \rightarrow 0 \\ k_2 \rightarrow 0}} \partial_{k_2} \partial_{\omega_1} \lim_{\substack{\omega_2 \rightarrow 0 \\ k_1 \rightarrow 0}} G^{xy|yz|0x} &= -\frac{1}{4}\lambda_2 + \frac{1}{2}\eta\tau_\Pi \\ \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} \partial_{k_1} \partial_{k_2} \lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G^{xy|0x|0y} &= -\frac{1}{4}\lambda_3.\end{aligned}\quad (13)$$

³ For a nice summary of the (r, a) notation see [18].

We will find that the leading order AdS/CFT computation of the correlators leads to the same expressions for $\lambda_1, \lambda_2, \lambda_3$ as those previously obtained by [1]:

$$\lambda_1 = \frac{N_c^2 T^2}{16}, \quad \lambda_2 = -\frac{N_c^2 T^2 \ln 2}{8}, \quad \lambda_3 = 0. \quad (14)$$

The paper is organized as follows. In Section 2 we derive the Kubo relations given in (13). More Kubo formulae can be found in Appendix B. In Section 3 we use AdS/CFT to evaluate the stress tensor correlators. We begin with a review of 2-point stress tensor correlators, and we comment how the hydrodynamic expansion of the retarded correlators and the AdS/CFT expressions are matched term-by-term in a gradient (small momenta) expansion. Then we present a similar result for the 3-point retarded stress tensor correlators which appear in (13), with the final result for the second-order hydro coefficients given by (14). Technical details are relegated to appendices. For example, the $\eta\tau_{II}$ contribution to (13) is derived in Appendix A. The graviton bulk-to-boundary propagators, expanded up to third order in momenta, are given in Appendix C. A few 2-point correlators are discussed in the text, but the rest of them are presented in Appendix D. Lastly, the on-shell (first, second and third order) gravitational action is given in Appendix E.

We use the following conventions: five-dimensional tensors, which live in the AdS-Schwarzschild geometry, have indices given by $K, L, M, \dots = 0, 1, 2, 3, 5 = t, x, y, z, u$; four-dimensional space-time indices are $\mu, \nu, \dots = 0, 1, 2, 3$; three-dimensional spatial indices are $i, j, \dots = 1, 2, 3 = x, y, z$. 4-vectors are denoted by bold letters $(\boldsymbol{x}, \boldsymbol{q})$, and 3-vectors with an arrow (\vec{x}) . We use a bar to denote background values, in both the field theory and in the holographic dual. E.g. $\bar{\epsilon}$ is the field theory background energy density, and \bar{g}_{MN} denotes the background AdS-Schwarzschild metric. For simplicity of notation, for the rest of the paper we will also work in units where $2\pi T = 1$.

2 Kubo formulae for λ_1, λ_2 and λ_3

2.1 The fluid velocity to leading order in the metric fluctuations and gradients

We begin by assuming that the fluid is initially in equilibrium in a flat space-time background, which is briefly distorted by a small gravitational perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \epsilon = \bar{\epsilon} + \delta\epsilon, \quad P = \bar{P} + \delta P \quad (15)$$

where $h_{\mu\nu}(t, z)$ is a small metric fluctuation, and $\delta\epsilon, \delta P$ are the induced energy density and pressure variations. In this section we will keep things general, and we will not enforce $\bar{P} = \frac{1}{3}\bar{\epsilon}$ until the end. We take the fluid to be initially at rest

$$U = \bar{U} + \delta U, \quad \bar{U}^\mu = (1, \vec{0}), \quad (16)$$

and so the background transverse projectors are simply

$$\bar{\Delta}^{ij} = \delta^{ij}, \quad \bar{\Delta}^{0i} = \bar{\Delta}^{00} = 0. \quad (17)$$

From the normalization condition $U^\mu U_\mu = -1$ we get the temporal component

$$U^0 = 1 + \frac{1}{2}h_{00} + O(h^2). \quad (18)$$

The spatial components U^i are determined from the stress tensor conservation law:

$$0 = \nabla_\mu T^{\mu\nu} = [\partial_\mu(\epsilon + P)]U^\mu U^\nu + (\partial_\mu P)g^{\mu\nu} + (\epsilon + P)\nabla_\mu(U^\mu U^\nu) + \nabla_\mu \Pi^{\mu\nu}, \quad (19)$$

where

$$\nabla_\mu(U^\mu U^\nu) = \partial_\mu(U^\mu U^\nu) + \Gamma_{\mu\sigma}^\mu U^\sigma U^\nu + \Gamma_{\mu\tau}^\nu U^\mu U^\tau. \quad (20)$$

To leading order in the metric fluctuations, this can be simplified using

$$[\partial_\mu(\epsilon + P)]U^\mu U^\nu + (\partial_\mu P)g^{\mu\nu} = \partial_\mu(\epsilon + P)\bar{U}^\mu \bar{U}^\nu + (\partial_\mu P)\eta^{\mu\nu} + O(h^2) \quad (21)$$

by noting that $\partial\epsilon$ and ∂P are $O(h)$.

First consider the $\nu = \perp \equiv x, y$ case and keep only terms through $O(h)$ in (19). Since $\bar{U}^\perp = 0$ and $\partial_\perp = 0$, the first two terms in (19) vanish at $O(h)$ by (21). Using (20), and noting that both U^\perp and Γ are $O(h)$,

$$\begin{aligned} \nabla_\mu(U^\mu U^\perp) &= \partial_\mu(\bar{U}^\mu U^\perp) + \Gamma_{\mu\tau}^\perp \bar{U}^\mu \bar{U}^\tau + O(h^2) \\ &= \dot{U}^\perp + \dot{h}_{0\perp} + O(h^2), \end{aligned} \quad (22)$$

where we use dots as shorthand for time derivatives. Now use the fact that

$$U_\perp = g_{\perp\nu}U^\nu = g_{\perp 0}U^0 + g_{\perp\perp}U^\perp + O(h^2) = h_{\perp 0} + U^\perp + O(h^2) \quad (23)$$

to rewrite this as

$$\nabla_\mu(U^\mu U^\perp) = \dot{U}_\perp + O(h^2). \quad (24)$$

Putting it all together (along with the fact that $\Pi^{\mu\nu}$ is $O(h)$ and so $\nabla\Pi \simeq \partial\Pi$), the transverse case of the conservation law (19) is

$$(\bar{\epsilon} + \bar{P})\dot{U}_\perp + \partial_\mu \Pi^{\mu\perp} = O(h^2). \quad (25)$$

This simplifies to

$$(\bar{\epsilon} + \bar{P})\dot{U}_\perp + \partial_z \Pi^{z\perp} = O(h^2), \quad (26)$$

since $\Pi^{\mu\nu}$ has only spatial components at $O(h)$ and ∂_\perp vanishes.

Our goal will be to work to leading order in frequencies ω and leading order in spatial momenta k , without assuming anything about the relative size of ω and $|k|$. (That is, we want to be able to handle $\omega \sim k^2$ and $\omega \sim |k|$ and $\omega = 0$ and $k = 0$ on an equal footing.) In (26), we have explicitly kept the term which is leading order in time derivatives of U . We now need the term that is leading order in spatial derivatives of U . That comes from the first-order hydro terms in $\partial_i \Pi^{i\perp}$.

So consider the first order hydro expansion of $\Pi^{\mu\nu}$:

$$\begin{aligned} \Pi_{1st}^{\mu\nu} &= -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\Delta^{\alpha\beta}\nabla_\alpha U_\beta \\ &= -\eta\Delta^{\mu\alpha}(\nabla_\alpha U_\beta + \nabla_\beta U_\alpha)\Delta^{\beta\nu} + (\tfrac{2}{3}\eta - \zeta)\Delta^{\mu\nu}\Delta^{\alpha\beta}\nabla_\alpha U_\beta, \end{aligned} \quad (27)$$

where ζ is the bulk viscosity and $\zeta = 0$ for a conformal theory. At first order in h ,

$$\Pi_{1st}^{ij} = -\eta(\nabla_i U_j + \nabla_j U_i) + (\tfrac{2}{3}\eta - \zeta)\delta_{ij}\nabla_k U_k + O(h^2), \quad (28)$$

and in particular

$$\Pi_{1st}^{z\perp} = -\eta(\partial_z U_\perp + \partial_0 h_{z\perp} - \partial_z h_{0\perp}) + O(h^2). \quad (29)$$

Putting this together with (26), we get

$$(\bar{\epsilon} + \bar{P})\dot{U}_\perp - \eta U_\perp'' - \eta \partial_z (\dot{h}_{z\perp} - h'_{0\perp}) + \partial_z (2\text{nd order hydro effects}) = O(h^2), \quad (30)$$

where primes are shorthand for z -derivatives. The solution is

$$U_\perp = \frac{D_\eta k(kh_{0\perp} + \omega h_{z\perp})}{-i\omega + D_\eta k^2} + O(\omega U_\perp, k U_\perp, h^2), \quad (31)$$

where

$$D_\eta \equiv \frac{\eta}{\bar{\epsilon} + \bar{P}} \quad (32)$$

is the diffusion constant associated with shear viscosity. Here $(\omega, 0, 0, k)$ is the four-momentum associated with the factors of h . The $O(\omega U_\perp, k U_\perp)$ corrections listed in (31) represent corrections due to second-order hydro, which are suppressed by additional factors of momenta⁴. Note that U_\perp vanishes at $O(h)$ if one sets $h_{\mu\perp} \propto q_\mu = (-\omega, 0, 0, k)$.

This leaves us with U_z , which we will determine by solving the remaining $\nu = z, 0$ components of the stress tensor conservation law, to linear order in the metric fluctuation, and to leading order in gradients. First we compute

$$\Pi_{1st}^{zz} \simeq -\frac{4}{3}\eta \left[U_z' + \frac{1}{2}(\dot{h}_{zz} - \frac{1}{2}\dot{h}_{\perp\perp} - 2h'_{0z}) \right] - \zeta \left[U_z' + \frac{1}{2}(\dot{h}_{ii} - 2h'_{0z}) \right], \quad (33)$$

where $h_{\perp\perp} \equiv h_{xx} + h_{yy}$. Then $\nabla_\mu T^{\mu z} = 0$ yields

$$\dot{U}_z - \left(\frac{4}{3}D_\eta + \frac{\zeta}{\bar{\epsilon} + \bar{P}} \right) U_z'' \simeq -\frac{P'}{\bar{\epsilon} + \bar{P}} + \frac{1}{2}h'_{00} + \frac{2}{3}D_\eta(\dot{h}_{zz} - \frac{1}{2}\dot{h}_{\perp\perp} - 2h'_{0z}) + \frac{\zeta}{2(\bar{\epsilon} + \bar{P})}(\dot{h}_{ii} - 2h'_{0z}), \quad (34)$$

while from $\nabla_\mu T^{\mu 0} = 0$ we get

$$U_z' \simeq h'_{0z} - \frac{1}{2}\dot{h}_{ii} - \frac{\dot{\epsilon}}{\bar{\epsilon} + \bar{P}}. \quad (35)$$

Specialize now to conformal theories ($\zeta = 0, \bar{P} = \frac{1}{3}\bar{\epsilon}$). If we stay away from the sound pole $\omega^2 \simeq \frac{1}{3}k^2$, then to leading order in derivatives we can ignore Π_{1st}^{zz} and the D_η terms in (34), and combine (34) and (35) to get

$$U_z \simeq \frac{1}{\partial_t^2 - \frac{1}{3}\partial_z^2} \left(\frac{1}{2}\dot{h}'_{00} + \frac{1}{6}\dot{h}'_{ii} - \frac{1}{3}h''_{0z} \right). \quad (36)$$

and

$$\epsilon = \bar{\epsilon} + \frac{4}{3} \frac{\bar{\epsilon}}{\partial_t^2 - \frac{1}{3}\partial_z^2} (\dot{h}'_{0z} - \frac{1}{2}h''_{00} - \frac{1}{2}\ddot{h}_{ii}) + \mathcal{O}(h^2, \omega h, kh). \quad (37)$$

In what follows we will derive Kubo relations for the second-order hydrodynamic coefficients from the response of the stress tensor component T^{xy} . From (1) and (2),

$$T^{xy} = \bar{\epsilon} \left[\frac{4}{3}U^{(1)x}U^{(1)y} + \frac{1}{3}(-h_{xy} + h_{x\mu}\eta^{\mu\nu}h_{\nu y}) \right] + \frac{1}{3}(\delta^{(1)}\epsilon)(-h_{xy}) + \Pi^{xy} + O(h^3), \quad (38)$$

where the superscript (1) denotes a contribution of order h and $\epsilon = \bar{\epsilon} + \delta^{(1)}\epsilon + \dots$.

⁴Our notation $O(\omega U_\perp, k U_\perp)$ assumes that the leading order in U_\perp is not zero.

2.2 λ_1

In the hydrodynamic expansion, λ_1 is the shear tensor-squared coefficient:

$$(\Pi^{\mu\nu})_{\lambda_1} = \lambda_1 \sigma^{\langle\mu} \sigma^{\nu\rangle\lambda} \quad (39)$$

where we will use the notation $(\dots)_{\lambda_1}$ to indicate that we are only showing terms of \dots that depend on λ_1 . We will also need

$$\sigma_{\perp z} = \sigma^{\perp z} + \mathcal{O}(h^2) \simeq (U'_\perp - h'_{0\perp}) + \dot{h}_{\perp z}, \quad (40)$$

and

$$\sigma_{xx} + \sigma_{yy} \simeq \frac{2}{3}(\frac{1}{2}\dot{h}_{\perp\perp} - \dot{h}_{zz}) - \frac{4}{3}(U'_z - h'_{0z}) \quad (41)$$

$$\sigma_{xy} \simeq \dot{h}_{xy}. \quad (42)$$

The λ_1 dependence $(T^{xy})_{\lambda_1}$ of (38) comes solely from the $(\Pi^{xy})_{\lambda_1}$ term at the order shown:

$$\begin{aligned} (T^{xy})_{\lambda_1} &\simeq (\Pi^{xy})_{\lambda_1} \simeq \lambda_1 \left[\sigma_{xy}(\sigma_{xx} + \sigma_{yy}) + \sigma_{xz}\sigma_{yz} \right] \\ &\simeq \lambda_1 \left[\frac{2}{3}\dot{h}_{xy}(\frac{1}{2}\dot{h}_{\perp\perp} - \dot{h}_{zz} - 2(U'_z - h'_{0z})) + (U'_x - h'_{0x} + \dot{h}_{xz})(U'_y - h'_{0y} + \dot{h}_{yz}) \right]. \end{aligned} \quad (43)$$

We now want to devise a simple Kubo-like formula for extracting λ_1 from some limit of a retarded 3-point correlator. To that end, note that in the limit of \vec{x} -independent sources, $h_{\mu\nu} = h_{\mu\nu}(t)$, equation (43) simplifies to

$$(T^{xy})_{\lambda_1} \simeq \lambda_1 \left[\frac{2}{3}\dot{h}_{xy}(\frac{1}{2}\dot{h}_{\perp\perp} - \dot{h}_{zz}) + \dot{h}_{xz}\dot{h}_{yz} \right]. \quad (44)$$

In momentum space this becomes

$$\begin{aligned} (T^{xy}(\mathbf{q}))_{\lambda_1} &\simeq -\lambda_1 \int d^4\mathbf{q}_1 \int d^4\mathbf{q}_2 \frac{1}{(2\pi)^4} \delta^4(\mathbf{q} - \mathbf{q}_1 - \mathbf{q}_2) \delta^3(\vec{q}_1) \delta^3(\vec{q}_2) \omega_1 \omega_2 \\ &\times \left[\frac{2}{3}h_{xy}(\mathbf{q}_1)(\frac{1}{2}h_{\perp\perp}(\mathbf{q}_2) - h_{zz}(\mathbf{q}_2)) + h_{xz}(\mathbf{q}_1)h_{yz}(\mathbf{q}_2) \right] \end{aligned} \quad (45)$$

where the 4-momentum $\mathbf{q} = (\omega, 0, 0, k)$ is associated with the measurement point, and the 4-momenta $\mathbf{q}_1 = (\omega_1, 0, 0, k_1)$, $\mathbf{q}_2 = (\omega_2, 0, 0, k_2)$ are associated with each one of the h factors respectively. By (11), this produces contributions to $G^{xy|xy|\perp\perp}$, $G^{xy|xy|zz}$ and $G^{xy|xz|yz}$ in the limit where all three spatial momenta vanish. For the purposes of our later AdS/CFT calculation, we will find that the least technically challenging case to calculate will be $G^{xy|xz|yz}$. So here we will focus on finding a Kubo-like formula for λ_1 in terms of $G^{xy|xz|yz}$. More Kubo-like formulas can be found in Appendix B. Consolidating identical terms in (11) gives

$$\begin{aligned} \langle T^{xy}(\mathbf{q}) \rangle_h &= \int d^4\mathbf{q}_1 \int d^4\mathbf{q}_2 \frac{1}{(2\pi)^4} \delta^4(\mathbf{q} - \mathbf{q}_1 - \mathbf{q}_2) G^{xy|xz|yz}(\mathbf{q}; -\mathbf{q}_1, -\mathbf{q}_2) h_{xz}(\mathbf{q}_1) h_{yz}(\mathbf{q}_2) \\ &+ \text{other } G\text{'s}. \end{aligned} \quad (46)$$

In what follows we will suppress writing the momentum dependence of the 3-point correlator, with the understanding that the momenta $-\mathbf{q}_1, -\mathbf{q}_2$ are associated with the second pair and third pair of indices of the 3-point correlator respectively. We are now ready to compare (46) to (45) to extract λ_1 :

$$\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G^{xy|xz|yz} = -\lambda_1 + (\lambda_1\text{-independent terms}). \quad (47)$$

We leave the derivation of the λ_1 -independent terms to Appendix A. The result is:

$$\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G^{xy|xz|yz} = -\lambda_1 + \eta\tau_{\Pi}. \quad (48)$$

2.3 λ_2

To evaluate the λ_2 contribution to the stress tensor T^{xy} , we need to compute

$$(T^{xy})_{\lambda_2} \simeq (\Pi^{xy})_{\lambda_2} \simeq \lambda_2 \frac{1}{2} \delta^{xl} \delta^{ym} (\sigma_{ln} \Omega_m^n + \sigma_{mn} \Omega_l^n), \quad (49)$$

Assuming, as we did before, that the fluctuations depend only on t, z coordinates, vorticity is given by

$$\Omega_i^j \simeq \Omega^{ij} \simeq \frac{1}{2} (\partial_i U_j - \partial_j U_i) = \frac{1}{2} \begin{pmatrix} 0 & 0 & -U'_x \\ 0 & 0 & -U'_y \\ U'_x & U'_y & 0 \end{pmatrix}. \quad (50)$$

Substituting (50) and (40) into (49) we find

$$(T^{xy})_{\lambda_2} \simeq \frac{1}{4} \lambda_2 \left(-U'_x (U'_y - h'_{0y} + \dot{h}_{yz}) - U'_y (U'_x - h'_{0x} + \dot{h}_{xz}) \right). \quad (51)$$

Again, we are interested in finding a simple Kubo relation which is amenable to a straightforward AdS/CFT computation. We settle on using $G^{xy|yz|0x}$ and we consider sources such that $h_{yz} = h_{yz}(t)$ and $h_{0x} = h_{0x}(z)$. After a substitution of U_{\perp} from (31) into (51), and a comparison with the relevant terms from (11)

$$\begin{aligned} \langle T^{xy}(\mathbf{q}) \rangle_h &= \int d^4 \mathbf{q}_1 \int d^4 \mathbf{q}_2 \frac{1}{(2\pi)^4} \delta^4(\mathbf{q} - \mathbf{q}_1 - \mathbf{q}_2) G^{xy|yz|0x}(\mathbf{q}; -\mathbf{q}_1, -\mathbf{q}_2) h_{yz}(\mathbf{q}_1) h_{0x}(\mathbf{q}_2) \\ &+ \text{other } G\text{'s} \end{aligned} \quad (52)$$

we are led to

$$\lim_{\substack{\omega_1 \rightarrow 0 \\ k_2 \rightarrow 0}} \partial_{k_2} \partial_{\omega_1} \lim_{\substack{\omega_2 \rightarrow 0 \\ k_1 \rightarrow 0}} G^{xy|yz|0x} = -\frac{1}{4} \lambda_2 + \frac{1}{2} \eta\tau_{\Pi}, \quad (53)$$

where the λ_2 -independent term $\frac{1}{2} \eta\tau_{\Pi}$ is derived in Appendix A. Other Kubo-type relations can be found in Appendix B.

2.4 λ_3

λ_3 is the coefficient of the square of the vorticity tensor in the hydrodynamic expansion of the stress tensor. To get λ_3 , we rely on computing the same component T^{xy} :

$$(T^{xy})_{\lambda_3} \simeq (\Pi^{xy})_{\lambda_3} \simeq \frac{1}{4}\lambda_3 U'_x U'_y \quad (54)$$

Substituting (31) we get

$$(T^{xy})_{\lambda_3} \simeq -\frac{1}{4}\lambda_3 k_1 k_2 \frac{D_\eta k_1 (k_1 h_{0x} + \omega_1 h_{zx})}{(-i\omega_1 + D_\eta k_1^2)} \frac{D_\eta k_2 (k_2 h_{0y} + \omega_2 h_{zy})}{(-i\omega_2 + D_\eta k_2^2)}. \quad (55)$$

A simple way to extract λ_3 from terms up to second order in momenta in the 3-point correlator (at higher order in momenta we would need to expand the stress tensor to terms that include higher order hydro coefficients) is to take the static limit: $\omega_1 = \omega_2 = 0$. Then, a comparison of (55) with the stress tensor response in terms of Green's functions (11), namely

$$\langle T^{xy}(\mathbf{q}) \rangle_h = \int d^4 \mathbf{q}_1 \int d^4 \mathbf{q}_2 \frac{1}{(2\pi)^4} \delta^4(\mathbf{q} - \mathbf{q}_1 - \mathbf{q}_2) G^{xy|0x|0y}(\mathbf{q}; -\mathbf{q}_1, -\mathbf{q}_2) h_{0x}(\mathbf{q}_1) h_{0y}(\mathbf{q}_2) + \text{other } G\text{'s}, \quad (56)$$

yields

$$\lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} \partial_{k_1} \partial_{k_2} \lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G^{xy|0x|0y} = -\frac{1}{4}\lambda_3. \quad (57)$$

We will use AdS/CFT to evaluate λ_3 from (57). However, for the reader's convenience we give other Kubo formulae in Appendix B.

3 Stress tensor correlators via AdS/CFT

The generating functional for the stress tensor correlators of $\mathcal{N}=4$ super Yang-Mills theory in the limit of large number of colors ($N_c \gg 1$) and at strong coupling is the on-shell five-dimensional gravitational action composed of the Einstein-Hilbert action, a cosmological constant term, the Gibbons-Hawking term, and holographic renormalization counterterms [20, 21, 22, 23]:

$$\mathcal{S} = \frac{N_c^2}{8\pi^2 R_{\text{AdS}}^3} \left[\int_{\mathcal{M}} \sqrt{-g} (R - 2\Lambda) + 2 \int_{\partial\mathcal{M}} \sqrt{-g_{\text{bdy}}} K + a \int_{\partial\mathcal{M}} \sqrt{-g_{\text{bdy}}} - \frac{R_{\text{AdS}}}{d-2} \int_{\partial\mathcal{M}} \sqrt{-g_{\text{bdy}}} R_{\text{bdy}} \right] \quad (58)$$

where the values of the cosmological constant and of the volume counterterm parameter a are

$$\Lambda = -\frac{d(d-1)}{2R_{\text{AdS}}^2} \quad \text{and} \quad a = -\frac{2(d-1)}{R_{\text{AdS}}}, \quad \text{with } d = 4. \quad (59)$$

The trace of the extrinsic curvature tensor can be expressed in terms of the induced metric on the boundary $(g_{\text{bdy}})_{MN} = g_{MN} - n_M n_N$ and the unit normal to the boundary n^M as

$$K = (g_{MN} - n_M n_N) \nabla^M n^N. \quad (60)$$

In general, \mathcal{M} asymptotes to an AdS space of radius R_{AdS} . Since we are studying finite-temperature super Yang-Mills theory, the background \mathcal{M} is five-dimensional AdS-Schwarzschild space [24]:

$$ds^2 = \frac{(\pi T R_{\text{AdS}})^2}{u} (-f(u) dt^2 + \vec{x}^2) + \frac{R_{\text{AdS}}^2 du^2}{4u^2 f(u)}, \quad f(u) = 1 - u^2. \quad (61)$$

Taking advantage of the fact that the AdS radius drops out of final results, it is convenient to set $R_{\text{AdS}} = 2$ and work in units $2\pi T = 1$ so that

$$ds^2 = \frac{-f(u) dt^2 + d\vec{x}^2}{u} + \frac{du^2}{u^2 f(u)}. \quad (62)$$

Next we will evaluate the on-shell action, by expanding the metric in fluctuations δg_{MN} (where $M, N = 0, 1, 2, 3, 5 = t, x, y, z, u$) around the AdS-Schwarzschild background. In imaginary time, AdS-Schwarzschild is smooth and non-singular, and 3-point correlators are given by Witten diagrams as depicted in Fig. 1a. In real time, as discussed in [25, 15], the corresponding diagrams for retarded correlators G_{raa} live in the right-quadrant of the Penrose diagram due to causality. The lines in this diagram represent advanced and retarded bulk-to-boundary propagators. These propagators are identified as solutions δg_M^N to the linearized equations of motion, which approach prescribed values at the boundary $u = 0$ and which reduce (up to gauge terms) to purely incoming/outgoing waves: $e^{-i\omega t}(1 - u)^{\pm i\omega/2}$.

3.1 Two-point stress tensor correlators and comparison with hydrodynamic expansion

First we will warm up with 2-point retarded correlators. We recall that the gravity fluctuations are taken to be independent of x, y coordinates, and are slowly varying functions of t, z . Given the symmetry of the problem, the fluctuations can be classified according to their transformations under rotations about the z axis as: $SO(2)$ tensors (δg_{xy} and $\delta g_{xx} - \delta g_{yy}$), vectors ($\delta g_{0x}, \delta g_{zx}$ and $\delta g_{0y}, \delta g_{zy}$), and scalars (all others).

For completeness we list in Appendix C the bulk metric fluctuations in momentum space as a series expansion in ω, k , and in terms of the boundary metric fluctuations, h_μ^ν . The retarded bulk-to-boundary graviton propagators are easily obtained from the expressions given in Appendix C by differentiating with respect to the boundary fields. We work in the gauge

$$\delta g_{M5} = 0. \quad (63)$$

The tensor modes δg_x^y and $\delta g_x^x - \delta g_y^y$ have propagators which do not have singularities for $\omega, k \ll 1$ [19]. The vector (or shear) modes, e.g. $\delta g_0^x, \delta g_z^x$, have poles typical of diffusion, and the scalar mode propagators have a sound pole [26]. We will refer to the vector metric fluctuations as shear modes, and to the scalar metric fluctuations as sound modes.

The computation of the stress tensor 1-point function is reviewed in Appendix E.1⁵:

$$\langle T^{00} \rangle_0 = 2 \frac{\delta \mathcal{S}}{\delta h_{00}} = \frac{3N_c^2}{2^7 \pi^2} \equiv \bar{\epsilon} \quad (64)$$

⁵Restoring the units, the energy density and pressure equal $\bar{\epsilon} = \frac{3\pi^2 N_c^2 T^4}{8}$, $\bar{P} = \frac{\pi^2 N_c^2 T^4}{8}$.

gives the energy density of the finite temperature strongly coupled super Yang-Mills theory and

$$\langle T^{ij} \rangle_0 = 2 \frac{\delta \mathcal{S}}{\delta h_{ij}} = \delta^{ij} \frac{N_c^2}{2^7 \pi^2} \equiv \delta^{ij} \bar{P}, \quad i, j = 1, 2, 3 = x, y, z, \quad (65)$$

gives its pressure.

The two-point retarded correlators are computed using the quadratic gravitational vertex reviewed in Appendix E.2. Since we are interested in ra correlators, one of the bulk-to-boundary propagators is retarded (the one associated with the largest time point on the boundary), and the other propagator is advanced. However, we should remember that the bulk-to-boundary propagators in momentum space are defined as retarded or advanced relative to the momentum conjugate to the boundary space-time point. 4-momentum conservation gives $\mathbf{q} = -\mathbf{q}_1$, where \mathbf{q} is the momentum associated with the point which has the largest time. This effectively transforms the advanced bulk-to-boundary propagator $\mathcal{G}_{\text{ar}}(\mathbf{q}_1, u)$ into a retarded propagator $\mathcal{G}_{\text{ra}}(\mathbf{q}, u)$.

From the decoupled fluctuation δg_y^x one recovers⁶ the stress tensor retarded 2-point function [2]

$$\begin{aligned} G_{\text{AdS}}^{xy|xy} &= -\frac{\delta^2 \mathcal{S}}{\delta^2 h_{xy}} \\ &= \frac{N_c^2}{2^7 \pi^2} - i \frac{N_c^2 \omega}{2^6 \pi^2} + \frac{(\omega^2(1 - \ln 2) - k^2) N_c^2}{2^6 \pi^2} + \dots \end{aligned} \quad (66)$$

The result derived in the hydrodynamic limit from solving (8) and (9) is

$$G_{\text{hydro}}^{xy|xy} = \frac{1}{3} \bar{\epsilon} - i \eta \omega + \eta \tau_{\Pi} \omega^2 - \frac{1}{2} \kappa (\omega^2 + k^2) + \dots \quad (67)$$

where dots denote terms which are higher order in gradients (and hydrodynamic expansion coefficients). By identifying (66) and (67) one obtains a handful of hydro coefficients, of first and second order⁷:

$$\eta = \frac{N_c^2}{2^6 \pi^2}, \quad \kappa = \frac{N_c^2}{2^5 \pi^2}, \quad \eta \tau_{\Pi} = \frac{N_c^2 (2 - \ln 2)}{2^6 \pi^2}. \quad (68)$$

From (64) and (68), the diffusion constant of the shear modes in the strongly coupled $\mathcal{N}=4$ super Yang-Mills plasma is

$$D_{\eta} = \frac{\eta}{\bar{\epsilon} + \bar{P}} = \frac{1}{2}. \quad (69)$$

This is the maximal set of hydrodynamic coefficients which can be determined from 2-point stress tensor correlators in the AdS-Schwarzschild background [2]. On the other hand, having found $\bar{\epsilon}$, \bar{P} , η , τ_{Π} and κ , we should be able to match all the other 2-point correlators obtained via AdS/CFT with their hydrodynamic expressions.

To see how this plays out we consider the 2-point functions of shear modes. We perform an expansion assuming a shear dispersion relation $\omega \sim k^2 \sim \lambda^2 \ll 1$, where λ is a small expansion

⁶For the sake of brevity, we suppress energy-momentum conservation delta functions.

⁷Restoring the units, this reads $\eta = \frac{\pi N_c^2 T^3}{8}$, $\kappa = \frac{N_c^2 T^2}{8}$, $\eta \tau_{\Pi} = \frac{N_c^2 (2 - \ln 2) T^2}{16}$.

parameter, and extend the results of Policastro, Son and Starinets [19] to second order in λ :

$$\begin{aligned}
G_{\text{AdS}}^{0x|0x} &= -\frac{\delta^2 \mathcal{S}}{\delta h_{0x} \delta h_{0x}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[-\frac{k^2 + 6i\omega}{2^4(-i\omega + \frac{1}{2}k^2)} + \frac{k^2(2\omega^2(\ln 2 - 1) - 2i\omega k^2 + k^4)}{2^3(-i\omega + \frac{1}{2}k^2)^2} + \dots \right] \\
G_{\text{AdS}}^{0x|xz} &= -\frac{\delta^2 \mathcal{S}}{\delta h_{0x} \delta h_{xz}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[-\frac{\omega k}{2^2(-i\omega + \frac{1}{2}k^2)} + \dots \right] \\
G_{\text{AdS}}^{xz|xz} &= -\frac{\delta^2 \mathcal{S}}{\delta h_{xz} \delta h_{xz}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{1}{2^3} - \frac{\omega^2}{2^2(-i\omega + \frac{1}{2}k^2)} + \dots \right].
\end{aligned} \tag{70}$$

As a result of this expansion, the shear poles are visible. The presence of the higher order poles in the 2-point correlators is an artifact of the expansion and indicates that the location $\omega = -i\frac{1}{2}k^2$ of the shear pole is shifted by higher order terms⁸.

This can be contrasted with

$$\begin{aligned}
G_{\text{hydro}}^{0x|0x} &= -\frac{\bar{\epsilon}(i\omega + \frac{1}{3}D_\eta k^2)}{-i\omega + D_\eta k^2} + \frac{\omega k^2(-(\eta\tau_\Pi - \frac{1}{2}\kappa)\omega + \frac{1}{2}i\kappa D_\eta k^2)}{(-i\omega + D_\eta k^2)^2} + \dots \\
G_{\text{hydro}}^{0x|xz} &= -\frac{\eta\omega k}{-i\omega + D_\eta k^2} + \dots \\
G_{\text{hydro}}^{xz|xz} &= \frac{1}{3}\bar{\epsilon} - \frac{\eta\omega^2}{-i\omega + D_\eta k^2} + \dots
\end{aligned} \tag{71}$$

There is an apparent mismatch in the hydrodynamic expansion and the AdS calculation in $G^{0x|0x}$, namely the coefficients ωk^4 and k^6 which multiply the second order shear pole disagree. This puzzle is resolved by noting that the expansion in small ω, k assuming a shear dispersion relation has the rather unwanted effect of mixing hydrodynamic coefficients of different order in each term of the series expansion: e.g. $\bar{\epsilon}$ and η at zeroth order in λ ; $\eta\tau_\Pi - \frac{1}{2}\kappa$ and $D_\eta\kappa$ at second order in λ ; etc. So a contamination with third-order hydro coefficients is to be expected, as forecast by the term proportional with $D_\eta\kappa$ which contains the product of two second order hydro coefficients. This is further elucidated by a comparison with what happens when instead expanding the correlators assuming $\omega \sim k \sim \lambda \ll 1$. Then, the AdS result is

$$\begin{aligned}
G_{\text{AdS}}^{0x|0x} &= \frac{N_c^2}{2^4 \pi^2} \left[\frac{3}{2^3} - i\frac{k^2}{2^2\omega} + \frac{k^2(2\omega^2(1 - \ln 2) - k^2)}{2^3\omega^2} + \dots \right] \\
G_{\text{AdS}}^{0x|xz} &= \frac{N_c^2}{2^4 \pi^2} \left[-i\frac{k}{2^2} + \frac{k(2\omega^2(1 - \ln 2) - k^2)}{2^3\omega} + \dots \right] \\
G_{\text{AdS}}^{xz|xz} &= \frac{N_c^2}{2^4 \pi^2} \left[\frac{1}{2^3} - i\frac{\omega}{2^2} + \frac{(2\omega^2(1 - \ln 2) - k^2)\pi^2 N_c^2 T^4}{2^3} + \dots \right]
\end{aligned} \tag{72}$$

⁸The shear pole, as extracted from second order hydro, is [2]: $\omega \simeq -iD_\eta k^2 - D_\eta\tau_\Pi\omega k^2$.

while the hydro expansion gives

$$\begin{aligned}
G_{\text{hydro}}^{0x|0x} &= \bar{\epsilon} - i\frac{\eta k^2}{\omega} + \frac{k^2(\eta\tau_{\Pi}\omega^2 - \frac{1}{2}\kappa\omega^2 - \eta D_{\eta}k^2)}{\omega^2} + \dots \\
G_{\text{hydro}}^{0x|xz} &= -i\eta k + \frac{k(\eta\tau_{\Pi}\omega^2 - \frac{1}{2}\kappa\omega^2 - \eta D_{\eta}k^2)}{\omega} + \dots \\
G_{\text{hydro}}^{xz|xz} &= \frac{1}{3}\bar{\epsilon} - i\eta\omega + \eta\tau_{\Pi}\omega^2 - \frac{1}{2}\kappa\omega^2 - \eta D_{\eta}k^2 + \dots
\end{aligned} \tag{73}$$

The ω poles are a remnant of expanding the shear pole, but now the two sets of 2-point correlators are perfectly matched, and at each term in the expansion there is no mixing between different order hydro coefficients. One may extract $\bar{\epsilon}, \eta$ and the combination $\eta\tau_{\Pi} - \frac{1}{2}\kappa$ with the same result as before. In this respect, the tensor mode 2-point correlator $G^{xy|xy}$ enables the identification of a larger set, since $\eta\tau_{\Pi}$ and κ can be obtained separately.

In appendix D we list the 2-point functions of sound modes, computed using a sound mode dispersion expansion $\omega \sim k \sim \lambda \ll 1$ to second order in λ , extending the results of Policastro, Son and Starinets [26]. The higher-order poles which appear in appendix D are also an artifact of expanding the higher-order attenuation terms which are present in the sound pole. There is no mixing of the different hydro coefficients at each term in the expansion, and the hydro and AdS results match, provided that one takes into account that the AdS computation yields tensor density correlators $(2(\delta^2\mathcal{S})/(\delta h_{\mu\nu}\delta h_{\rho\sigma})) = \delta\langle\sqrt{-g}T_{\mu\nu}\rangle/\delta h_{\rho\sigma}$ and not tensor correlators $(\delta\langle T^{\mu\nu}\rangle/\delta h_{\rho\sigma})$. The difference is a contact term, namely

$$G_{\text{hydro}}^{\mu\nu,\rho\sigma} = G_{\text{AdS}}^{\mu\nu,\rho\sigma} + \eta^{\rho\sigma}T^{\mu\nu}, \tag{74}$$

which arises from differentiating the volume factor $\sqrt{-g}$ with respect to the metric fluctuations.

3.2 λ_1, λ_2 and λ_3 via AdS/CFT

Due to the complexity of evaluating generic stress tensor 3-point functions, in this section we contend ourselves with computing $\lim_{k_1 \rightarrow 0, k_2 \rightarrow 0} G_{\text{AdS}}^{xy|yz|xz}$, $\lim_{\omega_1 \rightarrow 0, \omega_2 \rightarrow 0} G_{\text{AdS}}^{xy|0y|0x}$ and $\lim_{k_1 \rightarrow 0, \omega_2 \rightarrow 0} G_{\text{AdS}}^{xy|yz|0x}$. We leave a more complete computation of 3-point stress tensor correlators at arbitrary (albeit small) values of 4-momenta for future work.

Note that we are using 3-point correlators where one pair of legs, xy (which is associated with the largest-time boundary point) corresponds to tensor fluctuations, and the other leg pairs correspond to $O(2)$ vector (shear mode) fluctuations: $xz, x0$ and $yz, y0$. Because of the mixing between $\delta g_{\perp 0}$ and $\delta g_{\perp z}$, each of the corresponding bulk-to-boundary propagators is a 2×2 matrix. So, for example, the Witten diagram for $G^{xy|yz|xz}$ has four terms, depicted schematically in Fig. 2.

Moreover, each 2×2 shear-mode propagator contains a pure gauge (diffeomorphism) contribution whose behavior at the horizon is not in the form of an incoming/outgoing wave. This gauge artifact can (and does) cause technical difficulties in integrating the location of the vertex in the Witten diagram up to the horizon⁹. We will avoid this issue by computing correlators for which the diffeomorphism terms drop out, that is, we will focus on calculations where we can easily work with gauge-invariant modes.

⁹The problematic behavior is related to the fact that the diffeomorphism parameters (as constrained by the gauge condition $\delta g_{M5} = 0$) are non-analytic functions at the horizon (see equation (111)).

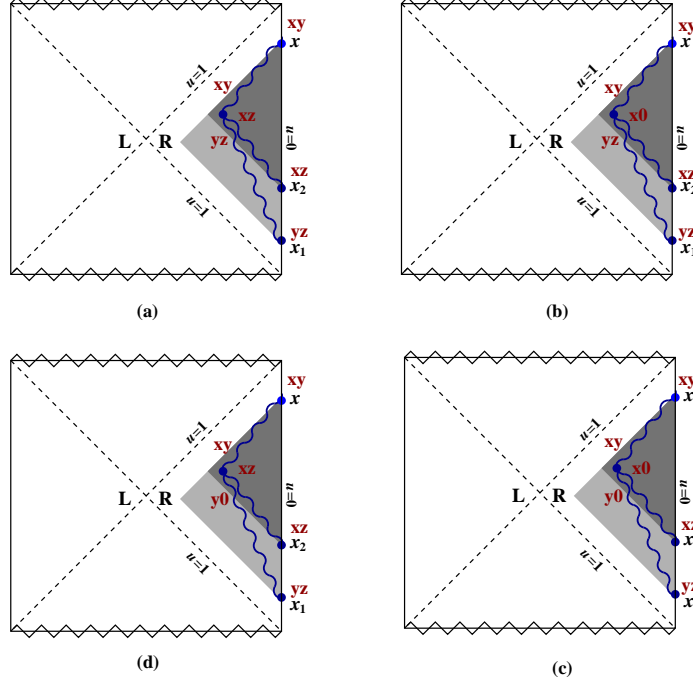


Figure 2: Witten diagrams for the retarded 3-point correlator $G^{xy|yz|xz}$ with the boundary point x having the largest time; x_1 and x_2 can have any time order.

As an example, the low momentum solution to the equation of motion for δh_0^x and δg_z^x which gives the retarded bulk-to-boundary propagators is¹⁰

$$\begin{aligned}
\delta g_0^x &= \tilde{C}_3(1-u)^{-i\omega/2} \left(i\frac{kf}{2} + \omega k \left(-\frac{f}{4} \ln(1+u) + \frac{\ln 2}{2} - \frac{u}{2} + \frac{(1-\ln 2)u^2}{2} \right) + \dots \right) + \tilde{D}_1 \\
\delta g_z^x &= \tilde{C}_3(1-u)^{-i\omega/2} \left(1 - i\frac{\omega}{2} \ln(1+u) + \omega^2 \left(-\frac{1}{2} \text{Li}_2\left(\frac{1-u}{2}\right) + \frac{1}{8} \ln^2(1+u) \right. \right. \\
&\quad \left. \left. + \left(1 - \frac{\ln 2}{2}\right) \ln(1+u) \right) + \dots \right) - \tilde{D}_1 \frac{k}{\omega}
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
\tilde{C}_3 &= \left(\frac{k}{\omega} - i\frac{k^3}{2\omega^2} - \frac{k(\omega^4(6\ln^2 2 - \pi^2) + 12\omega^2 k^2 \ln 2 + 6k^4)}{24\omega^3} + \dots \right) \left(h_0^x + \frac{\omega}{k} h_z^x \right) \\
\tilde{D}_1 &= \left(1 - i\frac{k^2}{2\omega} - \frac{k^2(2\omega^2 \ln 2 + k^2)}{4\omega^2} + \dots \right) h_0^x \\
&\quad + \left(-i\frac{k}{2} - \frac{k(2\omega^2 \ln 2 + k^2)}{4\omega} + \dots \right) h_z^x.
\end{aligned} \tag{76}$$

The \tilde{D}_1 terms in (76) represent a gauge mode associated with shifts in the x direction. One possible way to trivially isolate the gauge mode is to take the $k \rightarrow 0$ limit. Then the shear modes

¹⁰Here, since we are interested in sending the spatial momentum to 0 first, we give the shear mode propagators assuming a sound mode dispersion relation. In Appendix C we assumed a diffusion dispersion relation to highlight the presence of the diffusion pole.

decouple. $\delta g_0^x(k=0)$ is equal to the boundary value up to $O(\omega^3)$ corrections, and $\delta g_z^x(k=0)$ is a purely incoming wave at the horizon to the same $O(\omega^3)$ order:

$$\begin{aligned}\delta g_0^x(k=0) &= h_0^x(k=0) + \dots, \\ \delta g_z^x(k=0) &= h_z^x(k=0) \times \left(1 - \frac{\omega^2(6 \ln^2 2 - \pi^2)}{24} + \dots\right) (1-u)^{-i\omega/2} \left(1 - i\frac{\omega}{2} \ln(1+u)\right. \\ &\quad \left.+ \omega^2\left(-\frac{1}{2}\text{Li}_2\left(\frac{1-u}{2}\right) + \frac{1}{8} \ln^2(1+u) + \left(1 - \frac{\ln 2}{2}\right) \ln(1+u)\right) + \dots\right).\end{aligned}\quad (77)$$

As a result, only Fig. 2a contributes to $G_{\text{AdS}}^{xy|xz|yz}(k_1=0, k_2=0)$ [a consequence of $O(3)$ rotation invariance when $k=0$], and we expect no subtleties coming from the near horizon region.

Except for the specific form of the cubic gravitational vertex, the evaluation of the 3-point stress tensor correlator is no different than evaluating a causal scalar 3-point correlator with a derivative cubic vertex in the AdS-Schwarzschild bulk¹¹. Consider this scalar example, with cubic interaction $\int_{\mathcal{M}} \sqrt{-g} g^{MN} \partial_M \phi \partial_N \phi$, for the sake of simplicity of discussion, and examine the small frequency (with k 's zero) behavior of this contribution to the action \mathcal{S} . As we'll see, this is dominated by the near-horizon ($u \rightarrow 1$) contribution

$$\sqrt{-g} g^{uu} \phi \partial_u \phi \partial_u \phi. \quad (78)$$

The small frequency behavior of the retarded scalar bulk-to-boundary propagator is $(1-u)^{-i\omega/2}$. Therefore, computing the small frequency behavior of the integrand in (78) gives a contribution to the causal retarded correlator which in the near-horizon region is proportional to

$$\int du (1-u)^{-i(\omega_1+\omega_2)-1} (\omega_1 \omega_2 - (\omega_1 + \omega_2)^2) \quad (79)$$

where the terms quadratic in ω 's arise from taking the u derivatives. In arriving at (79) we took the momenta flowing through the a legs of the raa correlator to be ω_1 and ω_2 . By energy conservation, the momentum flowing through the r leg is $\omega = -(\omega_1 + \omega_2)$. The potential divergence at the horizon is regularized by the fact that the causal 3-point function of interest corresponds to $\omega_1 \rightarrow \omega_1 - i\epsilon$, $\omega_2 \rightarrow \omega_2 - i\epsilon$ and $\omega = -(\omega_1 + \omega_2) \rightarrow -(\omega_1 + \omega_2) + 2i\epsilon$.¹² Integrating (79) from $u=0$ to 1 yields

$$\frac{i}{\omega_1 + \omega_2} (\omega_1 \omega_2 - (\omega_1 + \omega_2)^2) \quad (80)$$

to leading order in ω 's. Counting powers of frequency as $\lambda \sim \omega_1 \sim \omega_2$, the near-horizon divergence of the integrand has enhanced the naively $O(\lambda^2)$ order of the integrand to an $O(\lambda)$ contribution to the 3-point function.

A similar story emerges for the $G_{\text{AdS}}^{xy|xz|yz}(k_1=0, k_2=0)$ correlator. There, the integrand, as given by the expansion of the cubic gravitational action, behaves near the horizon as

$$\frac{3N_c^2}{2^6} (1-u)^{-i(\omega_1+\omega_2)} \left(\frac{(\omega_1 + \omega_2)^2}{(1-u)} - i \frac{\ln 2 (\omega_1 + \omega_2)^3}{(1-u)} + \dots \right) \quad (81)$$

¹¹If the scalar supergravity fields have a derivative coupling of the type $\int_{\mathcal{M}} \sqrt{-g} g^{MN} \partial_M \phi \partial_N \phi$, then, for on-shell fields this reduces to $(m^2/2) \int_{\mathcal{M}} \sqrt{-g} \phi^3$ plus a total derivative term. However, in the text, we refrain from performing the integration by parts and merely comment on the behavior of the integrand near the horizon.

¹² See the discussion in sec II.B of [25], and see [27, 28].

where at least one of the ω factors arise from derivatives on the wave factor $(1 - u)^{-i(\omega_1 + \omega_2)}$.¹³ Regularizing the wave factors as before, and performing the integral over the black hole bulk up to the horizon, gives a finite contribution to the $G_{\text{AdS}}^{xy|xz|yz}(k_1=0, k_2=0)$ correlator:

$$\lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G_{\text{AdS}}^{xy|yz|xz} = \frac{N_c^2}{2^4 \pi^2} \left[\frac{1}{2^3} - i \frac{\omega_1 + \omega_2}{2^2} - \frac{(\omega_1 \omega_2 + \omega_1^2 + \omega_2^2)(\ln 2 - 1)}{2^2} + \dots \right]. \quad (82)$$

The last step is to identify the AdS/CFT correlator with the corresponding hydro expansion¹⁴

$$\lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G_{\text{hydro}}^{xy|yz|xz} = \frac{1}{3} \bar{\epsilon} - i\eta(\omega_1 + \omega_2) + \eta\tau_\Pi(\omega_1^2 + \omega_2^2 + \omega_1 \omega_2) - \frac{1}{2} \kappa(\omega_1^2 + \omega_2^2) - \lambda_1 \omega_1 \omega_2 + \dots \quad (83)$$

to get λ_1 :

$$\lambda_1 = \frac{N_c^2}{2^6 \pi^2}. \quad (84)$$

For the remaining λ_2, λ_3 coefficients we will need to take the limit $\omega \rightarrow 0$ of the shear modes. From Appendix C we get that $\delta g_z^x(\omega=0)$ is equal to its boundary value to order $O(k^3)$, and $\delta g_0^x(\omega=0)$ decouples:

$$\begin{aligned} \delta g_0^x(\omega=0) &= \left(1 - u^2 - k^2 u(1 - u) + \dots \right) h_0^x(\omega=0) \\ \delta g_z^x(\omega=0) &= h_z^x(\omega=0) + \dots \end{aligned} \quad (85)$$

Thus, once again, the graviton shear modes decouple, and the correlators needed to evaluate λ_2 and λ_3 are given by a single Witten-type diagram, with the causal graviton propagators being diagonal in both the tensor and shear mode sectors. The integral over the cubic action is straightforward and gives:

$$\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{AdS}}^{xy|0y|0x} = \frac{N_c^2}{2^6 \pi^2} \left[-\frac{1}{2} + (k_1^2 + k_2^2) + \dots \right]. \quad (86)$$

Comparison with the hydro expansion

$$\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{xy|0y|0x} = -\frac{1}{3} \bar{\epsilon} + \frac{1}{2} \kappa(k_2^2 + k_1^2) - \frac{1}{4} \lambda_3 k_1 k_2 + \dots \quad (87)$$

yields

$$\lambda_3 = 0. \quad (88)$$

¹³ As a result, finding the integrand in (81) only requires knowing the solution $\delta g^{(1)}$ to the linearized equations of motion to $O(\omega_1^2, \omega_2^2)$, as given in (79).

¹⁴ Given the discussion preceding (74), readers may wonder why we are identifying tensor density correlators (G_{AdS}) with tensor correlators (G_{hydro}). There is no difference for the particular components we are interested. In general,

$$\begin{aligned} G_{\text{hydro}}^{\mu\nu|\rho\sigma|\tau\zeta}(\mathbf{x}, \mathbf{y}, \mathbf{z}) &= G_{\text{AdS}}^{\mu\nu|\rho\sigma|\tau\zeta}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + G_{\text{AdS}}^{\mu\nu|\rho\sigma}(\mathbf{x}, \mathbf{y}) g^{\tau\zeta} \delta(\mathbf{x} - \mathbf{z}) + G_{\text{AdS}}^{\mu\nu|\tau\zeta}(\mathbf{x}, \mathbf{z}) g^{\rho\sigma} \delta(\mathbf{x} - \mathbf{y}) \\ &+ G_{\text{AdS}}^{\mu\nu}(\mathbf{x}) (g^{\rho\sigma} g^{\tau\zeta} + g^{\rho\tau} g^{\sigma\zeta} + g^{\rho\zeta} g^{\sigma\tau}) \delta(\mathbf{x} - \mathbf{y}) \delta(\mathbf{x} - \mathbf{z}). \end{aligned}$$

Similarly, we compute

$$\lim_{\substack{k_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{AdS}}^{xy|yz|0x} = \frac{N_c^2}{2^6 \pi^2} \omega_1 k_2 + \dots \quad (89)$$

which can be identified with

$$\lim_{\substack{k_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{xy|yz|0x} = (-\frac{1}{4} \lambda_2 + \frac{1}{2} \eta \tau_{\Pi}) \omega_1 k_2 + \dots \quad (90)$$

to yield¹⁵,

$$\lambda_2 = -\frac{N_c^2}{2^5 \pi^2} \ln 2. \quad (91)$$

This concludes our derivation of the second order hydrodynamic coefficients λ_1 , λ_2 and λ_3 from 3-point stress tensor correlators¹⁶.

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A $\eta \tau_{\Pi}$ contribution to $G_{\text{hydro}}^{xy|xz|yz}$ and $G_{\text{hydro}}^{xy|yz|0x}$

In this section we are interested in computing the $\eta \tau_{\Pi}$ contribution to T_{xy} to second order in gradients, and in the presence of a gravitational perturbation which includes only the shear modes: h_{x0} , h_{xz} , h_{y0} and h_{yz} . We begin with

$$(\Pi^{\mu\nu})_{\eta\tau_{\Pi}} = \eta \tau_{\Pi} \left(\langle U \nabla \sigma^{\mu\nu} \rangle + \frac{1}{3} \nabla \cdot U \sigma^{\mu\nu} \right) \quad (92)$$

¹⁵See [29] for arguments regarding a universal relationship between λ_1 , λ_2 and $\eta \tau_{\Pi}$.

¹⁶In addition to the computation detailed in the main text, we have performed a separate check of (14) using a different set of gauge-invariant modes e.g. choosing the boundary fields such that $h_x^x = -h_y^y$ or $h_0^0 = -\frac{1}{3}k^2(h_x^x + h_y^y) + \dots$, $h_z^z = (-1 + k^2)(h_x^x + h_y^y) + \dots$, $h_0^z = -\frac{1}{2}(ik + k\omega \ln 2)(h_x^x + h_y^y) + \dots$ where ellipsis denotes that we have required that the gauge component of the bulk-to-boundary propagators vanishes to order $O(\lambda^3)$ in a small $\omega \sim k \sim \lambda \ll 1$ expansion. For example, solving the hydrodynamic equations we compute

$$G_{\text{hydro}}^{xx-yy|0x|0x}(\omega_1=\omega_2=0) = -\frac{2}{3}\bar{\epsilon} + \kappa(k_1^2 + k_2^2) - \frac{1}{2}\lambda_3 k_1 k_2 + \dots,$$

$$G_{\text{hydro}}^{xx+yy-2zz|xy|xy}(k_1=k_2=0) = \frac{4}{3}\bar{\epsilon} - 4i\eta(\omega_1 + \omega_2) + 4\eta\tau_{\Pi}(\omega_1^2 + \omega_2^2 + \omega_1\omega_2) - 2\kappa^2(\omega_1^2 + \omega_2^2) - 4\lambda_1\omega_1\omega_2 + \dots,$$

$$G_{\text{hydro}}^{xx-yy|zx|0x}(k_1=0, \omega_2=0) = (-\frac{1}{2}\lambda_2 + \eta\tau_{\Pi})\omega_1 k_2 + \dots.$$

The corresponding AdS 3-point functions are

$$G_{\text{AdS}}^{xx-yy|0x|0x}(\omega_1=\omega_2=0) = -\frac{N_c^2}{2^6 \pi^2} + \frac{N_c^2}{2^5 \pi^2}(k_1^2 + k_2^2) + \dots,$$

$$G_{\text{AdS}}^{xx+yy-2zz|xy|xy}(k_1=k_2=0) = \frac{N_c^2}{2^5 \pi^2} - i\frac{N_c^2}{2^4 \pi^2}(\omega_1 + \omega_2) - \frac{N_c^2(\ln 2 - 1)}{2^4 \pi^2}(\omega_1^2 + \omega_2^2 + \omega_1\omega_2) + \dots,$$

and

$$G_{\text{AdS}}^{xx-yy|zx|0x}(k_1=0, \omega_2=0) = \frac{N_c^2}{2^5 \pi^2} \omega_1 k_2 + \dots.$$

where

$$\langle U \nabla \sigma^{\mu\nu} \rangle = \Delta^{\mu\rho} \Delta^{\nu\sigma} U^\zeta \nabla_\zeta \sigma_{\rho\sigma} - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} U^\zeta \nabla_\zeta \sigma_{\rho\sigma}. \quad (93)$$

Given the set of metric perturbations considered, $\sigma^{0\mu} = O(h^2)$. The non-vanishing components of σ^{ij} , to linear order in the shear metric fluctuations, are σ^{xz} and σ^{yz} given in (40) (where we recall that the shear tensor σ is symmetric). So for $(\Pi^{xy})_{\eta\tau\Pi}$ only the first term in (92) is non-vanishing.

We will also need the projectors $\Delta^{\mu\nu}$ to order $O(h^2)$:

$$\Delta^{\mu\nu} = \begin{pmatrix} 0 & U_x & U_y & 0 \\ U_x & 1 & 0 & -h_{xz} \\ U_y & 0 & 1 & -h_{yz} \\ 0 & -h_{xz} & -h_{yz} & 1 \end{pmatrix} + O(h^2) \quad (94)$$

where U_x, U_y are given in (31). Given that Δ^{xy} vanishes to $O(h^2)$, only the first term in (93) for $\mu = x, \nu = y$ contributes to terms quadratic in the metric fluctuations:

$$\partial_0 \sigma_{xy}^{(2)} - (\Gamma_{0x}^z \sigma_{yz} + \Gamma_{0y}^z \sigma_{xz}) + (\Delta^{xz} \partial_0 \sigma_{yz} + \Delta^{yz} \partial_0 \sigma_{xz}). \quad (95)$$

The last piece we must compute is σ_{xy} to second order in the fluctuations

$$\sigma_{xy}^{(2)} = h_{x\mu} \sigma^{(1)\mu y} + h_{\mu y} \sigma^{(1)x\mu} + \sigma^{(2)xy} \quad (96)$$

where $\sigma^{(2)xy}$ is the truncation of (4) to second order in fluctuations. Using that the second order expansion of the Christoffel symbols is

$$\Gamma_{\mu\nu}^{(2)\lambda} = -g^{\lambda\rho} h_{\rho\sigma} \Gamma_{\mu\nu}^{(1)\sigma}, \quad (97)$$

the shear tensor $\sigma_{xy}^{(2)}$ evaluates to

$$\sigma_{xy}^{(2)} = \partial_0 (U_x U_y) + \text{other h's}. \quad (98)$$

Substituting everything into (92) we arrive at

$$\begin{aligned} (\Pi^{xy})_{\eta\tau\Pi} &= \eta\tau\Pi \left(\partial_0^2 (U_x U_y) - \frac{1}{2} (\dot{h}_{zx} - h'_{0x}) (U'_y - h'_{0y} + \dot{h}_{zy}) - \frac{1}{2} (\dot{h}_{zy} - h'_{0y}) (U'_x - h'_{0x} + \dot{h}_{zx}) \right. \\ &\quad \left. - h_{xz} (\dot{U}'_y - \dot{h}'_{0y} + \ddot{h}_{zy}) - h_{yz} (\dot{U}'_x - \dot{h}'_{0x} + \ddot{h}_{zx}) + \text{other h's} \right). \end{aligned} \quad (99)$$

Recalling that U_\perp velocities vanish for z -independent metric perturbations (that is, U_\perp vanishes at zero spatial momentum), we can now extract the desired $\eta\tau\Pi$ contribution to the correlators $G_{\text{hydro}}^{xy|xz|yz}(k_1=0, k_2=0)$ and $G_{\text{hydro}}^{xy|yz|0x}(k_1=0, \omega_2=0)$:

$$G_{\text{hydro}}^{xy|xz|yz}(k_1=0, k_2=0) = \eta\tau\Pi (\omega_1 \omega_2 + \omega_1^2 + \omega_2^2) + \dots + \eta\tau\Pi\text{-independent terms} \quad (100)$$

$$G_{\text{hydro}}^{xy|yz|0x}(k_1=0, \omega_2=0) = \frac{1}{2} \eta\tau\Pi k_2 \omega_1 + \dots + \eta\tau\Pi\text{-independent terms}. \quad (101)$$

More general explicit computations show that there are no other second-order hydro coefficients contributions to (48, 53, 57) besides the ones we explicitly went over in Section 2 and in this appendix.

B More Kubo formulae

Using the method outlined in Section 2 one can derive a variety of Kubo relations. We give below a few more to supplement (48), (53) and (57). To simplify notation we suppress writing the momentum dependence of a retarded raa 3-point correlator $G^{\dots|\dots|\dots}(\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2; -\mathbf{q}_1, -\mathbf{q}_2)$, with the understanding that the momenta $-\mathbf{q}_1, -\mathbf{q}_2$ are associated with the second pair and third pair of indices of the 3-point correlator respectively.

B.1 λ_1

$$\begin{aligned}
\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G_{\text{hydro}}^{xy|xy|xx} &= -\frac{2}{3}\lambda_1 - \frac{1}{2}\kappa + \frac{5}{3}\eta\tau_{\Pi} \\
\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G_{\text{hydro}}^{xy|xy|zz} &= \frac{4}{3}\lambda_1 - \frac{1}{2}\kappa - \frac{1}{3}\eta\tau_{\Pi} \\
\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G_{\text{hydro}}^{xx|xy|xy} &= -\frac{2}{3}\lambda_1 - \frac{1}{6}\kappa + \eta\tau_{\Pi} \\
\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G_{\text{hydro}}^{zz|xy|xy} &= \frac{4}{3}\lambda_1 - \frac{1}{6}\kappa - \eta\tau_{\Pi} \\
\lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} \partial_{\omega_1} \partial_{\omega_2} \lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} G_{\text{hydro}}^{yz|xy|xz} &= -\lambda_1 + \eta\tau_{\Pi}.
\end{aligned} \tag{102}$$

Noting that $O(3)$ rotational symmetry is restored in the limit of vanishing spatial momenta, we can use (102) together with $O(3)$ covariance to compute other correlators. E.g. $G^{xx|xy|xy}(k_1=0, k_2=0) = G^{xx|xz|xz}(k_1=0, k_2=0) = G^{zz|xz|xz}(k_1=0, k_2=0)$ and $G^{zz|xy|xy}(k_1=0, k_2=0) = G^{yy|xz|xz}(k_1=0, k_2=0)$. Rotational symmetry can also be used to check the last Kubo relation in (102): a $\pi/4$ rotation in the (y, z) plane can be used to show that $G^{yz|xy|xz}(k_1=0, k_2=0) = \frac{1}{2}[G^{yy|xz|xz}(k_1=0, k_2=0) - G^{yy|xy|xy}(k_1=0, k_2=0)]$.

B.2 λ_2

The order of limits ($\omega_1 \rightarrow 0$ and $k_2 \rightarrow 0$) is important here (due to the presence of a sound pole in the correlators). On the one hand,

$$\begin{aligned}
\lim_{k_2 \rightarrow 0} \partial_{k_2} \lim_{\omega_1 \rightarrow 0} \partial_{\omega_1} \lim_{\substack{k_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{xx|xz|0x} &= -\lambda_2 - \kappa + 2\eta\tau_{\Pi} \\
\lim_{k_2 \rightarrow 0} \partial_{k_2} \lim_{\omega_1 \rightarrow 0} \partial_{\omega_1} \lim_{\substack{k_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{yy|xz|0x} &= -\frac{1}{2}\lambda_2 - \kappa + \eta\tau_{\Pi}
\end{aligned} \tag{103}$$

but a different order of limits yields

$$\begin{aligned}
\lim_{\omega_1 \rightarrow 0} \partial_{\omega_1} \lim_{k_2 \rightarrow 0} \partial_{k_2} \lim_{\substack{k_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{xx|xz|0x} &= -\frac{1}{2}\lambda_2 - \frac{1}{2}\kappa + \frac{4}{3}\eta\tau_\Pi + \frac{\eta^2}{2\bar{\epsilon}} \\
\lim_{\omega_1 \rightarrow 0} \partial_{\omega_1} \lim_{k_2 \rightarrow 0} \partial_{k_2} \lim_{\substack{k_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{zz|xz|0x} &= \frac{1}{2}\lambda_2 + \frac{1}{2}\kappa - \frac{2}{3}\eta\tau_\Pi - \frac{\eta^2}{\bar{\epsilon}} \\
\lim_{\omega_1 \rightarrow 0} \partial_{\omega_1} \lim_{k_2 \rightarrow 0} \partial_{k_2} \lim_{\substack{k_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{yz|xy|0x} &= \frac{1}{4}\lambda_2 + \frac{1}{2}\kappa - \frac{1}{2}\eta\tau_\Pi - \frac{3\eta^2}{4\bar{\epsilon}}.
\end{aligned} \tag{104}$$

B.3 λ_3

$$\begin{aligned}
\lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} \partial_{k_1} \partial_{k_2} \lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{xx|0y|0y} &= \frac{1}{2}\lambda_3, \\
\lim_{\substack{k_1 \rightarrow 0 \\ k_2 \rightarrow 0}} \partial_{k_1} \partial_{k_2} \lim_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow 0}} G_{\text{hydro}}^{00|0x|0x} &= \frac{1}{2}\lambda_3.
\end{aligned} \tag{105}$$

In the relation (53) given in the main text, one gets the same answer independent of the order of limits $\omega_1 \rightarrow 0$ and $k_2 \rightarrow 0$.

C Retarded bulk-to-boundary propagators

The metric fluctuations, solutions to the linearized Einstein equations, are in general coupled. We work in the gauge

$$\delta g_{M5} = 0, \quad M = 0, 1, 2, 3, 5 = t, x, y, z, u. \tag{106}$$

We define

$$\delta g_N^M \equiv \bar{g}^{MP} \delta g_{PN} \tag{107}$$

The fluctuations considered in this paper are independent of x, y coordinates. Relative to the rotation group $SO(2)$ about the z axis, one distinguishes tensor fluctuations ($\delta g_{xy}; \delta g_{xx} - \delta g_{yy}$), vector fluctuations ($\delta g_{0t}, \delta g_{xz}$; and $\delta g_{y0}, \delta g_{yz}$, and scalar fluctuations ($\delta g_{00}, \delta g_{zz}, \delta g_{0z}$, and $\delta g_{xx} + \delta g_{yy}$). The tensor fluctuations are completely decoupled, while the vector and scalar fluctuations all mix within their respective sectors for generic ω, k . For the vector, δg_{x0} mixes with δg_{xz} and δg_{y0} mixes with δg_{yz} .

Since we are interested in the hydrodynamic regime, where the fields vary slowly with t, z , the equations of motion are solved in the bulk perturbatively in ω, k . The bulk-to-boundary causal propagator of coupled fluctuations will contain one term which behaves like an incoming/outgoing wave at the horizon, in addition to terms which are diffeomorphism terms. The existence of these diffeomorphism terms is inferred by solving the equations of motion near the horizon to leading order, substituting an ansatz of the type $F(u)(1-u)^r$ as in [26]. The values of $r = \pm i\omega/2$ correspond to the incoming/outgoing waves; the other possible values of the exponent (e.g. $r = 0, -1/2$) correspond to the diffeomorphism terms. Lastly, we require that the bulk fields approach prescribed values at the $u = 0$ boundary: $\delta g_M^N \xrightarrow{u \rightarrow 0} h_\mu^\nu$.

We find¹⁷

$$\begin{aligned}
\delta g_0^0 &= C_1(1-u)^{-i\omega/2} \left(-\frac{2k^2(1-u)}{3} + \dots \right) + D_0 - D_5 \frac{f - uf' + 2\omega^2 u}{\sqrt{f}} \\
\delta g_x^x &= C_1(1-u)^{-i\omega/2} \left(1 - i\frac{\omega}{2} \ln(1+u) + \omega^2 \left(-\frac{1}{2} \text{Li}_2\left(\frac{1-u}{2}\right) + \frac{1}{8} \ln^2(1+u) \right. \right. \\
&\quad \left. \left. + \ln(1+u) \left(1 - \frac{1}{2} \ln 2 \right) \right) + k^2 \left(\frac{1}{3} \ln(1+u) + \frac{2}{3} (1-u) \right) + \dots \right) \\
&\quad + C_2(1-u)^{-i\omega/2} \left(1 - i\frac{\omega}{2} \ln(1+u) + \omega^2 \left(-\frac{1}{2} \text{Li}_2\left(\frac{1-u}{2}\right) + \frac{1}{8} \ln^2(1+u) \right. \right. \\
&\quad \left. \left. + \ln(1+u) \left(1 - \frac{1}{2} \ln 2 \right) \right) - k^2 \ln(1+u) + \dots \right) - D_5 \sqrt{f} \\
\delta g_y^y &= C_1(1-u)^{-i\omega/2} \left(1 - i\frac{\omega}{2} \ln(1+u) + \omega^2 \left(-\frac{1}{2} \text{Li}_2\left(\frac{1-u}{2}\right) + \frac{1}{8} \ln^2(1+u) \right. \right. \\
&\quad \left. \left. + \ln(1+u) \left(1 - \frac{1}{2} \ln 2 \right) \right) + k^2 \left(\frac{1}{3} \ln(1+u) + \frac{2}{3} (1-u) \right) + \dots \right) \\
&\quad - C_2(1-u)^{-i\omega/2} \left(1 - i\frac{\omega}{2} \ln(1+u) + \omega^2 \left(-\frac{1}{2} \text{Li}_2\left(\frac{1-u}{2}\right) + \frac{1}{8} \ln^2(1+u) \right. \right. \\
&\quad \left. \left. + \ln(1+u) \left(1 - \frac{1}{2} \ln 2 \right) \right) - k^2 \ln(1+u) + \dots \right) - D_5 \sqrt{f} \\
\delta g_z^z &= C_1(1-u)^{-i\omega/2} \left(-2 + i\omega \ln(1+u) + \omega^2 \left(\text{Li}_2\left(\frac{1-u}{2}\right) - \frac{1}{4} \ln^2(1+u) \right. \right. \\
&\quad \left. \left. - (2 - \ln 2) \ln(1+u) \right) + k^2 \left(-\frac{2}{3} \ln(1+u) + \frac{2}{3} (1-u) \right) + \dots \right) \\
&\quad - D_3 \frac{2k}{\omega} + D_5 (-\sqrt{f} + 2k^2 \arcsin(u)) \\
\delta g_0^z &= C_1(1-u)^{-i\omega/2} \left(-ikf + \omega k(1-u) \left(u + \frac{1}{2} (1+u) (\ln(1+u) - 2 \ln 2) + \dots \right) \right. \\
&\quad \left. + D_3 + D_0 \frac{kf}{2\omega} - D_5 \omega k (u \sqrt{f} + \arcsin(u)) \right) \\
\delta g_0^x &= C_3(1-u)^{-i\omega/2} k \left(-if + \frac{\omega}{2} f \ln \frac{1+u}{2} + (\omega + ik^2) u(1-u) + \dots \right) + D_1 \frac{\omega}{k} \\
\delta g_0^y &= C_4(1-u)^{-i\omega/2} k \left(-if + \frac{\omega}{2} f \ln \frac{1+u}{2} + (\omega + ik^2) u(1-u) + \dots \right) + D_2 \frac{\omega}{k} \\
\delta g_z^x &= C_3(1-u)^{-i\omega/2} \left(-2 + i\omega \ln \frac{1+u}{2} + k^2 + \dots \right) - D_1 \\
\delta g_z^y &= C_4(1-u)^{-i\omega/2} \left(-2 + i\omega \ln \frac{1+u}{2} + k^2 + \dots \right) - D_2
\end{aligned}$$

¹⁷We recall that, for simplicity of notation, we work in units where $2\pi T = 1$. Alternatively, one should think of ω and k as energy and momentum made dimensionless by division with $2\pi T$.

$$\begin{aligned}\delta g_y^x &= C_5(1-u)^{-i\omega/2} \left(1 - i\frac{\omega}{2} \ln(1+u) + \omega^2 \left(-\frac{1}{2} \text{Li}_2\left(\frac{1-u}{2}\right) \right. \right. \\ &\quad \left. \left. + \frac{1}{8} \ln^2(1+u) + \left(1 - \frac{\ln 2}{2}\right) \ln(1+u) \right) - k^2 \ln(1+u) + \dots \right) \end{aligned} \quad (108)$$

where the coefficients $C_1, \dots, C_5, D_0, \dots, D_3, D_5$ are given in terms of the boundary fields as follows¹⁸:

$$\begin{aligned}C_1 &= \left(\frac{k^2}{3\omega^2 - k^2} - i\frac{2\omega k^4}{(3\omega^2 - k^2)^2} + \frac{k^2 A}{24(3\omega^2 - k^2)^3} + \dots \right) h_0^0 \\ &+ \left(\frac{\omega^2 - k^2}{2(3\omega^2 - k^2)} - i\frac{\omega k^2(\omega^2 - k^2)}{(3\omega^2 - k^2)^2} + \frac{(\omega^2 - k^2)A}{48(3\omega^2 - k^2)^3} + \dots \right) (h_x^x + h_y^y) \\ &+ \left(-\frac{\omega^2}{3\omega^2 - k^2} + i\frac{2\omega^3 k^2}{(3\omega^2 - k^2)^2} - \frac{\omega^2 A}{24(3\omega^2 - k^2)^3} + \dots \right) h_z^z \\ &+ \left(-\frac{2\omega k}{3\omega^2 - k^2} + i\frac{4\omega^2 k^3}{(3\omega^2 - k^2)^2} - \frac{\omega k A}{12(3\omega^2 - k^2)^3} + \dots \right) h_0^z \\ A &= 9\omega^6(\pi^2 - 6\ln^2 2) - 6\omega^4 k^2(\pi^2 + 24\ln 2 - 6\ln^2 2) + \omega^2 k^4(\pi^2 + 48\ln 2 - 6\ln^2 2) - 32k^6 \\ C_2 &= \left(\frac{1}{2} + \frac{\omega^2(\pi^2 - 6\ln^2 2)}{48} + \dots \right) (h_x^x - h_y^y) \\ C_3 &= \left(\frac{ik}{2(-i\omega + \frac{1}{2}k^2)} - \frac{\omega k((i\omega + \frac{1}{2}k^2)\ln 2 - k^2)}{4(-i\omega + \frac{1}{2}k^2)^2} + \dots \right) (h_0^x + \frac{\omega}{k} h_z^x) \\ C_4 &= \left(\frac{ik}{2(-i\omega + \frac{1}{2}k^2)} - \frac{\omega k((i\omega + \frac{1}{2}k^2)\ln 2 - k^2)}{4(-i\omega + \frac{1}{2}k^2)^2} + \dots \right) (h_0^y + \frac{\omega}{k} h_z^y) \\ C_5 &= \left(1 + \frac{\omega^2(\pi^2 - 6\ln^2 2)}{24} + \dots \right) h_y^x \end{aligned} \quad (109)$$

$$\begin{aligned}D_0 &= \left(\frac{3\omega^2}{3\omega^2 - k^2} - i\frac{2\omega k^4}{(3\omega^2 - k^2)^2} + \frac{2\omega^2 k^4(3\omega^2(2 - \ln 2) + k^2(\ln 2 - 4))}{(3\omega^2 - k^2)^3} + \dots \right) h_0^0 \\ &- \left(\frac{\omega^2}{3\omega^2 - k^2} + i\frac{\omega k^2(\omega^2 - k^2)}{(3\omega^2 - k^2)^2} - \frac{\omega^2 k^2(\omega^2 - k^2)(3\omega^2(2 - \ln 2) + k^2(\ln 2 - 4))}{(3\omega^2 - k^2)^3} + \dots \right) \\ &\times (h_x^x + h_y^y) \\ &+ \left(-\frac{\omega^2}{3\omega^2 - k^2} + i\frac{2\omega^3 k^2}{(3\omega^2 - k^2)^2} - \frac{2\omega^4 k^2(3\omega^2(2 - \ln 2) + k^2(\ln 2 - 4))}{(3\omega^2 - k^2)^3} + \dots \right) h_z^z \\ &+ \left(-\frac{2\omega k}{3\omega^2 - k^2} + i\frac{4\omega^2 k^3}{(3\omega^2 - k^2)^2} - \frac{4\omega^3 k^3(3\omega^2(2 - \ln 2) + k^2(\ln 2 - 4))}{(3\omega^2 - k^2)^3} + \dots \right) h_0^z \\ D_1 &= \left(-i\frac{k}{-i\omega + \frac{1}{2}k^2} + \frac{k^3(\omega \ln 2 + (i/2)k^2)}{2(-i\omega + \frac{1}{2}k^2)^2} + \dots \right) h_0^x \\ &+ \left(-\frac{k^2}{2(-i\omega + \frac{1}{2}k^2)} + \frac{k^2\omega(\omega \ln 2 + (i/2)k^2)}{2(-i\omega + \frac{1}{2}k^2)^2} + \dots \right) h_z^x \end{aligned}$$

¹⁸The appearance of higher order sound and diffusion poles is only an artefact of expanding in small ω, k of simple poles which have additional attenuation pieces. For example, C_3 can be repackaged as $k^2(1 - (i/2)\omega \ln 2 + \dots)/(k^2 - i\omega(2 - k^2) + \dots)$.

$$\begin{aligned}
D_2 &= \left(-i \frac{k}{-i\omega + \frac{1}{2}k^2} + \frac{k^3(\omega \ln 2 + (i/2)k^2)}{2(-i\omega + \frac{1}{2}k^2)^2} + \dots \right) h_0^y \\
&+ \left(-\frac{k^2}{2(-i\omega + \frac{1}{2}k^2)} + \frac{k^2\omega(\omega \ln 2 + (i/2)k^2)}{2(-i\omega + \frac{1}{2}k^2)^2} + \dots \right) h_z^y \\
D_3 &= \left(-\frac{3\omega k}{2(3\omega^2 - k^2)} + i \frac{3\omega^2 k^3}{(3\omega^2 - k^2)^2} + \frac{\omega k^3(9\omega^4 \ln 2 - 3\omega^2 k^2 \ln 2 + 2k^4)}{(3\omega^2 - k^2)^3} + \dots \right) h_0^0 \\
&+ \left(\frac{\omega k}{2(3\omega^2 - k^2)} + i \frac{3\omega^2 k(\omega^2 - k^2)}{2(3\omega^2 - k^2)^2} + \frac{\omega k(\omega^2 - k^2)(9\omega^4 \ln 2 - 3\omega^2 k^2 \ln 2 + 2k^4)}{2(3\omega^2 - k^2)^3} + \dots \right) \\
&\times (h_x^x + h_y^y) \\
&+ \left(\frac{\omega k}{2(3\omega^2 - k^2)} - i \frac{3\omega^4 k}{(3\omega^2 - k^2)^2} - \frac{\omega^3 k(9\omega^4 \ln 2 - 3\omega^2 k^2 \ln 2 + 2k^4)}{(3\omega^2 - k^2)^3} + \dots \right) h_z^z \\
&+ \left(\frac{3\omega^2}{3\omega^2 - k^2} - i \frac{6\omega^3 k^2}{(3\omega^2 - k^2)^2} - \frac{2\omega^2 k^2(9\omega^4 \ln 2 - 3\omega^2 k^2 \ln 2 + 2k^4)}{(3\omega^2 - k^2)^3} + \dots \right) h_0^3 \\
D_5 &= \left(\frac{k^2}{3\omega^2 - k^2} - i \frac{2\omega k^4}{(3\omega^2 - k^2)^2} + \frac{2k^4(9\omega^4(1 - \ln 2) + 3\omega^2 k^2(\ln 2 - 2) - k^4)}{3(3\omega^2 - k^2)^3} + \dots \right) h_0^0 \\
&- \left(\frac{\omega^2}{3\omega^2 - k^2} + i \frac{\omega k^2(\omega^2 - k^2)}{(3\omega^2 - k^2)^2} - \frac{k^2(\omega^2 - k^2)(9\omega^4(1 - \ln 2) + 3\omega^2 k^2(\ln 2 - 2) - k^4)}{3(3\omega^2 - k^2)^3} + \dots \right) \\
&\times (h_x^x + h_y^y) \\
&+ \left(-\frac{\omega^2}{3\omega^2 - k^2} + i \frac{2\omega^3 k^2}{(3\omega^2 - k^2)^2} - \frac{2\omega^2 k^2(9\omega^4(1 - \ln 2) + 3\omega^2 k^2(\ln 2 - 2) - k^4)}{3(3\omega^2 - k^2)^3} + \dots \right) h_z^z \\
&+ \left(-\frac{2\omega k}{3\omega^2 - k^2} + i \frac{4\omega^2 k^3}{(3\omega^2 - k^2)^2} - \frac{4\omega k^3(9\omega^4(1 - \ln 2) + 3\omega^2 k^2(\ln 2 - 2) - k^4)}{3(3\omega^2 - k^2)^3} + \dots \right) h_0^z
\end{aligned} \tag{110}$$

The diffeomorphism parameters which give rise to the diffeomorphism terms in the bulk-to-boundary propagators are:

$$\begin{aligned}
\xi_M(\omega, k, u) &= \left(-i \frac{D_0 f}{2\omega u} + i D_5 \omega \sqrt{f}, i \frac{D_1}{k u}, i \frac{D_2}{k u}, i \frac{D_3}{\omega u} - i \frac{D_5 k \arcsin(u)}{u}, \frac{D_5}{u \sqrt{f}} \right) \\
\xi^M(\omega, k, u) &= \left(i \frac{D_0}{2\omega} - i D_5 \frac{\omega u}{\sqrt{f}}, i \frac{D_1}{k}, i \frac{D_2}{k}, i \frac{D_3}{\omega} - i D_5 k \arcsin(u), D_5 \sqrt{f} u \right)
\end{aligned} \tag{111}$$

That is, the “ D ” terms in (108) are given by $(\delta g_N^M)_{\text{diff}} = \nabla_N \xi^M + \nabla^M \xi_N$. The D_0, D_1, D_2, D_3 terms in (111) generate boundary diffeomorphisms. The D_5 term generates an infinitesimal scale transformation in the vicinity of the $u = 0$ boundary, plus additional boundary diffeomorphisms to preserve the gauge condition $\delta g_{M5} = 0$. Note that these diffeomorphism parameters induce singular gauge transformations at the horizon. This is a consequence of working in the gauge $\delta g_{M5} = 0$.

D Sound mode 2-point correlators

From the coupled fluctuations $\delta g_x^x, \delta g_y^y, \delta g_z^x, \delta g_z^z, \delta g_0^0$ (where for simplicity of notation we lumped together the sound modes with one more fluctuation, namely $\delta g_x^x - \delta g_y^y$ which can actually be decoupled) and from the form of the on-shell quadratic gravitational action (see Appendix E) we get

$$\begin{aligned}
G_{\text{AdS}}^{xx|xx} &= -4 \frac{\delta^2 \mathcal{S}}{\delta h_{xx} \delta h_{xx}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{(7\omega^2 - k^2)}{2^3(3\omega^2 - k^2)} - i \frac{(3\omega^4 - 3\omega^2 k^2 + k^4)\omega}{(3\omega^2 - k^2)^2} \right. \\
&\quad + \frac{1}{2(3\omega^2 - k^2)^3} \left(\omega^2(18\omega^6(1 - \ln 2) + \omega^4 k^2(24 \ln 2 - 39) + \omega^2 k^4(28 - 12 \ln 2) \right. \\
&\quad \left. \left. + k^6(2 \ln 2 - 7) \right) \right) + \dots \left. \right]
\end{aligned}$$

$$\begin{aligned}
G_{\text{AdS}}^{xx|yy} &= -4 \frac{\delta^2 \mathcal{S}}{\delta h_{xx} \delta h_{yy}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{(\omega^2 + k^2)}{2^3(3\omega^2 - k^2)} + i \frac{(3\omega^4 - k^4)\omega}{2(3\omega^2 - k^2)^2} \right. \\
&\quad + \frac{1}{2(3\omega^2 - k^2)^3} \left(9\omega^8(\ln 2 - 1) + \omega^6 k^2(15 - 3 \ln 2) - \omega^4 k^4(8 + 3 \ln 2) \right. \\
&\quad \left. \left. + (3 + \ln 2)\omega^2 k^6 - k^8 \right) \right) + \dots \left. \right]
\end{aligned}$$

$$\begin{aligned}
G_{\text{AdS}}^{00,00} &= -4 \frac{\delta^2 \mathcal{S}}{\delta^2 h_{00}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{3(5k^2 - 3\omega^2)}{2^3(3\omega^2 - k^2)} - i \frac{3\omega k^4}{(3\omega^2 - k^2)^2} \right. \\
&\quad \left. + \frac{k^4(9\omega^4(1 - \ln 2) + 3\omega^2 k^2(\ln 2 - 2) - k^4)}{(3\omega^2 - k^2)^3} + \dots \right]
\end{aligned}$$

$$\begin{aligned}
G_{\text{AdS}}^{zz|zz} &= -4 \frac{\delta^2 \mathcal{S}}{\delta h_{zz} \delta h_{zz}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{(7\omega^2 - k^2)}{2^3(3\omega^2 - k^2)} - i \frac{3\omega^5}{(3\omega^2 - k^2)^2} \right. \\
&\quad \left. + \frac{\omega^4(9\omega^4(1 - \ln 2) + 3\omega^2 k^2(\ln 2 - 2) - k^4)\pi^2}{(3\omega^2 - k^2)^3} + \dots \right]
\end{aligned}$$

$$\begin{aligned}
G_{\text{AdS}}^{00|0z} &= -2 \frac{\delta^2 \mathcal{S}}{\delta h_{00} \delta h_{0z}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{3\omega k}{2(3\omega^2 - k^2)} \right. \\
&\quad \left. - i \frac{3\omega^2 k^3}{(3\omega^2 - k^2)^2} + \frac{\omega k^3(9\omega^4(1 - \ln 2) + 3\omega^2 k^2(\ln 2 - 2) - k^4)}{(3\omega^2 - k^2)^3} + \dots \right]
\end{aligned}$$

$$\begin{aligned}
G_{\text{AdS}}^{00|zz} &= -4 \frac{\delta^2 \mathcal{S}}{\delta h_{00} \delta h_{zz}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{3(\omega^2 + k^2)}{2^3(3\omega^2 - k^2)} - i \frac{3\omega^3 k^2}{(3\omega^2 - k^2)^2} \right. \\
&\quad \left. + \frac{\omega^2 k^2 (9\omega^4 (1 - \ln 2) + 3\omega^2 k^2 (\ln 2 - 2) - k^4)}{(3\omega^2 - k^2)^3} + \dots \right] \\
G_{\text{AdS}}^{0z|0z} &= - \frac{\delta^2 \mathcal{S}}{\delta h_{0z} \delta h_{0z}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{(9\omega^2 + k^2)}{2^3(3\omega^2 - k^2)} - i \frac{3\omega^3 k^2}{(3\omega^2 - k^2)^2} \right. \\
&\quad \left. + \frac{\omega^2 k^2 (9\omega^4 (1 - \ln 2) + 3\omega^2 k^2 (\ln 2 - 2) - k^4)}{(3\omega^2 - k^2)^3} + \dots \right] \\
G_{\text{AdS}}^{0z|zz} &= -2 \frac{\delta^2 \mathcal{S}}{\delta h_{0z} \delta h_{zz}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{\omega k}{2(3\omega^2 - k^2)} - i \frac{3\omega^4 k}{(3\omega^2 - k^2)^2} \right. \\
&\quad \left. + \frac{k\omega^3 (9\omega^4 (1 - \ln 2) + 3(\ln 2 - 2)\omega^2 k^2 - k^4)}{(3\omega^2 - k^2)^3} + \dots \right] \\
G_{\text{AdS}}^{00|xx} &= -4 \frac{\delta^2 \mathcal{S}}{\delta h_{00} \delta h_{xx}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{3(\omega^2 + k^2)}{2^3(3\omega^2 - k^2)} + i \frac{3\omega k^2 (\omega^2 - k^2)}{2(3\omega^2 - k^2)^2} \right. \\
&\quad \left. - \frac{k^2 (\omega^2 - k^2) (9\omega^4 (1 - \ln 2) + 3\omega^2 k^2 (\ln 2 - 2) - k^4)}{2(3\omega^2 - k^2)^3} + \dots \right] \\
G_{\text{AdS}}^{0z|xx} &= -2 \frac{\delta^2 \mathcal{S}}{\delta h_{0z} \delta h_{xx}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{\omega k}{2(3\omega^2 - k^2)} + i \frac{3(\omega^2 - k^2)k\omega^2}{2(3\omega^2 - k^2)^2} \right. \\
&\quad \left. - \frac{\omega k (\omega^2 - k^2) (9\omega^4 (1 - \ln 2) + 3(\ln 2 - 2)\omega^2 k^2 - k^4)}{2(3\omega^2 - k^2)^3} + \dots \right] \\
G_{\text{AdS}}^{zz|xx} &= -4 \frac{\delta^2 \mathcal{S}}{\delta h_{zz} \delta h_{xx}} \\
&= \frac{N_c^2}{2^4 \pi^2} \left[\frac{(\omega^2 + k^2)}{2^3(3\omega^2 - k^2)} + i \frac{3(\omega^2 - k^2)\omega^3}{2(3\omega^2 - k^2)^2} \right. \\
&\quad \left. + \frac{\omega^2 (k^2 - \omega^2) (9\omega^4 (1 - \ln 2) + 3(\ln 2 - 2)\omega^2 k^2 - k^4)}{2(3\omega^2 - k^2)^3} + \dots \right]
\end{aligned} \tag{112}$$

These 2-point functions exhibit, as expected, the sound pole¹⁹ expanded in small ω, k : $\omega \simeq \pm k/\sqrt{3} - i\frac{\eta k^2}{6P}$.

E On-shell gravitational action

For completeness, in this appendix we give the linear, quadratic and cubic gravitational vertex. This corresponds to expanding the gravitational action (58) in terms linear, quadratic and cubic in the linearized on-shell fluctuations:

$$\mathcal{S} = \bar{\mathcal{S}} + \delta^{(1)}\mathcal{S} + \delta^{(2)}\mathcal{S} + \delta^{(3)}\mathcal{S} + \dots \quad (113)$$

E.1 First order action

To linear order in the fluctuations (58) is a boundary term²⁰:

$$\delta^{(1)}\left(\int_{\mathcal{M}} \sqrt{-g}(R - 2\Lambda) + \int_{\partial\mathcal{M}} \sqrt{-g_{\text{bdy}}}(a + 2K)\right) = \frac{1}{2} \int_{\partial\mathcal{M}} \sqrt{-g_{\text{bdy}}} \left[n^5 \left(-(\partial_5 g_{\mu\nu}) \delta g^{\mu\nu} + (\partial_5 g_{\mu\nu}) g^{\mu\nu} \delta g^\rho_\rho \right) + a \delta g^\rho_\rho \right] \quad (114)$$

where $\mu, \nu, \rho = 0, 1, 2, 3$, and all indices are raised and lowered with the background metric, e.g. $\delta g^{\mu\nu} = \delta g_{\rho\sigma} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma}$. As expected, the Gibbons-Hawking term has contributed to the cancellation of the terms linear in derivatives of the metric fluctuations at the boundary. Also, the leading order divergence (proportional to $1/u_B^2$, where the boundary value is of the radial coordinate $u_B \rightarrow 0$) from the Einstein-Hilbert plus cosmological constant action and from the Gibbons-Hawking term is canceled by the boundary volume counterterm.

In the black hole background, the on-shell action linear in the boundary fields is finite [26]²¹:

$$\delta^{(1)}\mathcal{S} = \frac{N_c^2}{2^6 \pi^2} \int_{u=0} \left(-\frac{3}{4} \delta g_0^0 + \frac{1}{4} \sum_{i=1,2,3} \delta g_i^i \right) \quad (115)$$

where we have introduced the shorthand notation

$$\int_{u=0} \dots = \int d^4 \mathbf{x} \dots \Big|_{u=0}. \quad (116)$$

For example, the one-point function of the stress tensor T^{00} is obtained by differentiating with respect to $h_0^0 \equiv \delta g_0^0|_{u=0}$:

$$\langle T_0^0 \rangle_0 = 2 \frac{\delta \mathcal{S}}{\delta h_0^0} \Big|_{h_\mu^\nu=0} = 2 \frac{\delta(\delta^{(1)}\mathcal{S})}{\delta h_0^0} = -\frac{3N_c^2}{2^7 \pi^2}. \quad (117)$$

¹⁹The sound pole complete expression is [19] $v_s k - \frac{i}{2(\epsilon+P)}(\zeta + \frac{4}{3}\eta)k^2$ where ζ is the bulk viscosity, η is the shear viscosity, and the speed of sound is $v_s = \sqrt{\partial P/\partial \epsilon}$.

²⁰The second counterterm, proportional to the boundary Ricci scalar, does not contribute to this order.

²¹In AdS background, as opposed to the AdS-Schwarzschild case we consider, the on-shell action linear in fluctuations is zero.

To get the correct dimension for the stress tensor, we can restore the temperature dependence

$$\langle T_0^0 \rangle_0 = (2\pi T)^4 \times \left(-\frac{3N_c^2}{2^7\pi^2} \right) = -\frac{3}{8}\pi^2 N_c^2 T^4. \quad (118)$$

E.2 Second order action

Now expand (58) to second order in the linearized on-shell fluctuations. To highlight the role of the second counterterm in (58), and explain why one needs to go beyond the quadratic action given by [26], we first expand

$$\int_{\mathcal{M}} \sqrt{-g}(R - 2\Lambda) + \int_{\partial\mathcal{M}} \sqrt{-g_{\text{bdy}}}(a + 2K) \quad (119)$$

and collect the $(\delta g)^2$ terms. Since the fluctuations obey the linearized equations of motion, we are left again with a total derivative term (basically the first-order expansion of (114)). (The bulk term cancelled in the first-order expansion of the action (119) because the background solved the Einstein equations. The bulk term cancels now because the fluctuations are on-shell.) After these considerations, this is what is left:

$$\begin{aligned} & \frac{1}{4} \int_{u=0} \sqrt{-g_{\text{bdy}}} n^5 \left(-(\partial_5 \delta g_\nu^\mu) \delta g_\mu^\nu + (\partial_5 \delta g_\mu^\mu) \delta g_\nu^\nu \right) \\ & + \frac{1}{4} \int_{u=0} \sqrt{-g_{\text{bdy}}} \left[n^5 \left((\partial_5 g_{\mu\nu}) \delta g_\rho^\nu \delta g^{\mu\rho} - (\partial_5 g_{\mu\nu}) g^{\mu\nu} \delta g_\sigma^\rho \delta g_\rho^\sigma - \frac{1}{2} (\partial_5 g_{\mu\nu}) \delta g^{\mu\nu} \delta g_\rho^\rho \right. \right. \\ & \quad \left. \left. + \frac{1}{2} (\partial_5 g_{\mu\nu}) g^{\mu\nu} \delta g_\rho^\rho \delta g_\sigma^\sigma \right) + \frac{1}{2} a \delta g_\mu^\mu \delta g_\nu^\nu - a \delta g_\nu^\mu \delta g_\mu^\nu \right]. \end{aligned} \quad (120)$$

Again, the boundary volume counterterm, with its a coefficient, removes a $1/u^2$ divergence from the contact terms in (120), besides contributing to the finite terms. Upon substituting the black hole background metric (62), and

$$n^5 = -u\sqrt{1-u^2}, \quad \sqrt{-g_{\text{bdy}}} = \frac{\sqrt{1-u^2}}{u^2} \quad (121)$$

in the second order expansion (120) we arrive at

$$\begin{aligned} & \frac{1}{8} \int_{u=0} \frac{1}{u} \partial_5 \left(-\delta g_\mu^\mu \delta g_\nu^\nu + \delta g_\nu^\mu \delta g_\mu^\nu \right) \\ & + \frac{1}{4} \int_{u=0} \left(\frac{3}{4} (h_0^0)^2 - \frac{1}{2} h_0^0 h_i^i + h_i^0 h_0^i + \frac{1}{4} h_i^i h_j^j - \frac{1}{2} h_j^i h_i^j \right) \end{aligned} \quad (122)$$

where $\mu, \nu, \rho = 0, 1, 2, 3$; $i, j = 1, 2, 3$ and Einstein summation convention was used²².

However, while the action (119) is properly regularized and gives correct answers for the field theory stress tensor two-point functions up to terms linear in ω, k (and therefore suffices as long

²²This agrees with (3.15) in [26]. An apparent discrepancy is resolved by noting that the derivative terms in [26] include $\partial_5((h_0^3)^2)$, whereas we have $\partial_5(h_3^0 h_0^3)$. This difference is reflected in a different coefficient of $h_3^0 h_0^3$.

as one is interested in linear hydrodynamic coefficients), it diverges as $1/u$ to quadratic order in ω, k . To remove this divergence a second counterterm is needed, proportional to the boundary Einstein-Hilbert action [20, 21, 22, 23]:

$$- \frac{R_{\text{AdS}}}{d-2} \int_{\partial\mathcal{M}} \sqrt{-g_{\text{bdy}}} R_{\text{bdy}} \quad (123)$$

where we recall that for us $d = 4$ and we set $R_{\text{AdS}} = 2$. It is easy to see that this second counterterm contributes to leading order only to terms quadratic in ω, k ,

$$\frac{1}{2} \int_{u=0} \frac{\sqrt{1-u^2}}{u^2} \left(\partial^2 \delta g_\nu^\mu + \partial^\mu \partial_\nu \delta g_\rho^\rho - \partial_\rho \partial^\mu \delta g_\nu^\rho - \partial^\rho \partial_\nu \delta g_\rho^\mu - \delta_\nu^\mu (\partial^2 \delta g_\rho^\rho - \partial^\rho \partial_\sigma \delta g_\rho^\sigma) \right) \delta g_\mu^\nu \quad (124)$$

since the curvature tensor is quadratic in derivatives. In (124) indices are raised and lowered on the partial derivatives with the boundary metric $\bar{g}_{\text{bdy}\mu\nu} = \text{diag}(-f/u, 1/u, 1/u, 1/u)$. Since there is one inverse metric per term in (124) we see that the contribution coming from this counterterm will be divergent as $1/u$. The conclusion is that this counterterm's job, to second order in fluctuations and second order in ω, k , is only to remove divergences from (122), without any finite term subtraction. At fourth order in ω, k we get a finite contribution, coming from terms linear in u in δg_j^i . For completeness, we give below the leading order in ω, k of (124):

$$\begin{aligned} \int_{u=0} \frac{1}{u} \Big(& (\omega^2 - k^2)[(h_2^1)^2 - h_1^1 h_2^2] + (k^2 h_0^0 - h_3^3 \omega^2)(h_1^1 + h_2^2) \\ & - \omega k h_0^3 (h_1^1 + h_2^2) + \omega k (h_0^1 h_3^1 + h_0^2 h_3^2) + \omega^2 [(h_3^1)^2 + (h_3^2)^2] + k^2 [(h_0^1)^2 + (h_0^2)^2] \Big) \end{aligned} \quad (125)$$

The second-order on-shell gravitational action $\delta^{(2)}\mathcal{S}$ is given by the sum of equations (122) and (125) multiplied by the gravitational prefactor $N_c^2/(2^6 \pi^2)$.

E.3 Cubic action

In writing the cubic action, it is helpful to start with the second-order Einstein-Hilbert plus cosmological constant action

$$\begin{aligned} \delta^{(2)}\mathcal{S}_{\text{EH}} &\equiv \delta^{(2)} \int_{\mathcal{M}} \sqrt{-g} (R - 2\Lambda) \\ &= \frac{1}{2} \int_{\mathcal{M}} \sqrt{-\bar{g}} \nabla_M \left[\nabla^M \left(\frac{3}{4} \delta g^{KL} \delta g_{KL} - \frac{1}{4} \delta g_K^K \delta g_L^L \right) + \frac{1}{2} \nabla_N (\delta g_K^K \delta g^{MN}) \right. \\ &\quad \left. - 2 \nabla_N (\delta g^{MP} \delta g_P^N) + \delta g^{KN} \nabla_N \delta g_K^M + \delta g^{MN} \nabla_N \delta g_K^K \right] \end{aligned} \quad (126)$$

where the indices are being raised and lowered with the background metric \bar{g}_{MN} , and the derivatives are background covariant. In general, there is another contribution to the second-order action, which is proportional to the linearized equation of motion. Since δg_{MN} solves the linearized equation of motion, the only non-vanishing contribution is the total derivative given in (126). In

expanding to terms cubic in the fluctuations, the Einstein-Hilbert plus cosmological constant action receives contributions from $\delta^{(1)}(\delta^{(2)}\mathcal{S}_{\text{EH}})$ as well as from the expansion of the term proportional to the linearized equations of motion:

$$\delta^{(3)}\mathcal{S}_{\text{EH}} = \frac{1}{3}\delta^{(1)}\left(\delta^{(2)}\mathcal{S}_{\text{EH}}\right) + \frac{1}{3!}\delta^{(1)}\int_{\mathcal{M}}\sqrt{-g}\frac{1}{2}(L_{MN} - \frac{1}{2}g_{MN}L)\delta g^{MN}. \quad (127)$$

A brute force calculation gives

$$\begin{aligned} \frac{1}{3}\delta^{(1)}\left(\delta^{(2)}\mathcal{S}_{\text{EH}}\right) = \int_{\mathcal{M}}\sqrt{-\bar{g}}\nabla_M\bigg[& -\frac{1}{24}(\delta g_K^K)^2\nabla^M\delta g_L^L + \frac{1}{8}\delta g^{MN}\nabla_N(\delta g_K^K)^2 + \frac{1}{24}(\delta g_K^K)^2\nabla_N\delta g^{MN} \\ & + \frac{1}{8}\delta g_K^K\nabla^M(\delta g_{NP}\delta g^{NP}) + \frac{1}{12}\delta g_{NP}\delta g^{NP}\nabla^M\delta g_K^K - \frac{2}{3}\delta g^{MP}\delta g_P^N\nabla_N\delta g_K^K \\ & - \frac{1}{6}\delta g_K^K\delta g^{NP}\nabla_N\delta g_P^M - \frac{1}{3}\delta g_K^K\delta g^{MP}\nabla_N\delta g_P^N + \frac{1}{3}\delta g^{NP}\delta g_{PQ}\nabla_N\delta g^{MQ} \\ & + \delta g^{MP}\delta g_{PQ}\nabla_N\delta g^{NQ} - \frac{2}{3}\delta g^{MP}\delta g^{NQ}\nabla_N\delta g_{PQ} - \frac{1}{12}\delta g^{PQ}\delta g_{PQ}\nabla_N\delta g^{MN} \\ & - \frac{1}{3}\delta g^{MN}\nabla_N(\delta g^{PQ}\delta g_{PQ}) - \frac{2}{3}\delta g^{NP}\delta g_P^Q\nabla^M\delta g_{NQ}\bigg]. \end{aligned} \quad (128)$$

The L_{MN} tensor which appears in (127) was introduced in [30]²³:

$$\begin{aligned} -\frac{1}{2}L_{MN} & \equiv \delta^{(1)}R_{MN} - \frac{1}{2}\delta g_{MN}(\bar{R} - 2\Lambda) \\ & = -\frac{1}{2}(\nabla^K\nabla_K\delta g_{MN} + \nabla_M\nabla_N\delta g_K^K - \nabla^K\nabla_M\delta g_{NK} - \nabla^K\nabla_N\delta g_{MK}) + d\delta g_{MN}R_{\text{AdS}}^{-2} \end{aligned} \quad (129)$$

where we recall that the background Ricci scalar is $\bar{R} = 2(d+1)\Lambda/(d-1) = -d(d+1)R_{\text{AdS}}^{-2}$ and $d = 4$ for us. In terms of L_{MN} , the linearized equation of motion is $L_{MN} - \frac{1}{2}\bar{g}_{MN}L = 0$.

In [30] it was also shown that the second order variation of the equation of motion, $\frac{1}{2}\delta^{(1)}(L_{MN} - \frac{1}{2}\bar{g}_{MN}L)$ is expressed in terms of another tensor, V_{MN} :

$$\begin{aligned} -\frac{1}{2}\delta^{(1)}L_{MN} & \equiv V_{MN} = \delta g^{PQ}(\nabla_P\nabla_Q\delta g_{MN} + \nabla_M\nabla_N\delta g_{PQ} - \nabla_P\nabla_M\delta g_{NQ} - \nabla_P\nabla_N\delta g_{MQ}) \\ & - \frac{1}{2}(2\nabla_P\delta g^{PQ} - \nabla^Q\delta g_P^P)(\nabla_M\delta g_{NQ} + \nabla_N\delta g_{MQ} - \nabla_Q\delta g_{MN}) \\ & + \frac{1}{2}[(\nabla_M\delta g^{PQ})\nabla_N\delta g_{PQ} + 2(\nabla^P\delta g_M^Q)\nabla_P\delta g_{QN} - 2(\nabla^P\delta g_M^Q)\nabla_Q\delta g_{PN}]. \end{aligned} \quad (130)$$

Lastly, substituting into the second term of the cubic action (127), one finds [30]

$$\frac{1}{3!}\delta^{(1)}\int_{\mathcal{M}}\sqrt{-g}\frac{1}{2}(L_{MN} - \frac{1}{2}g_{MN}L)\delta g^{MN} = -\frac{1}{6}\int_{\mathcal{M}}\sqrt{-\bar{g}}(V_{MN} - \frac{1}{2}\bar{g}_{MN}V)\delta g^{MN}. \quad (131)$$

The total derivative terms omitted from the cubic action in [30] are those we give in (128). We do not throw away these terms, and keep their contribution (from both boundary and horizon) to the retarded momentum space 3-point stress tensor correlators.

²³ L_{MN} given in equation (2.6) in [30] has been further simplified by commuting two covariant derivatives, and subsequently using that the AdS curvature tensor is $R_{MPN}^Q = -(g_{MN}\delta_P^Q - g_{PN}\delta_M^Q)R_{\text{AdS}}^{-2}$. Arutyunov and Frolov have further set $R_{\text{AdS}} = 1$.

For the correlators we evaluate in Section 3.2, we only need cubic vertices of one tensor mode and two shear modes. This cuts down significantly the number of relevant terms from the cubic action (127).

The Gibbons-Hawking term contribution to the cubic action is most easily evaluated by making explicit use of the gauge condition $\delta g_{M5} = 0$ and of the form of the unit normal vector to the boundary (121):

$$\begin{aligned}
\delta^{(3)}\mathcal{S}_{\text{GH}} &= \frac{1}{3}\delta^{(1)}(\delta^{(2)}\mathcal{S}_{\text{GH}}) \\
&= \frac{1}{3}\delta^{(1)}\left[\int_{u=0}\sqrt{-g}\bar{g}^{55}\left((\delta g^{\mu\nu}-\frac{1}{2}g^{\mu\nu}\delta g_\rho^\rho)\partial_5\delta g_{\mu\nu}-(\delta g_\rho^\mu\delta g^{\nu\rho}-\frac{1}{2}\delta g_\rho^\rho\delta g^{\mu\nu})\partial_5g_{\mu\nu}\right.\right. \\
&\quad \left.\left.+\frac{1}{4}(\delta g^{\mu\nu}\delta g_{\mu\nu}-\frac{1}{2}(\delta g_\mu^\mu)^2)g^{\rho\sigma}\partial_5g_{\rho\sigma}\right)\right] \\
&= \int_{u=0}\sqrt{-g}\bar{g}^{55}\left[-\left(\delta g_\rho^\mu\delta g^{\nu\rho}-\frac{1}{2}\delta g^{\mu\nu}\delta g_\rho^\rho-\frac{1}{4}\bar{g}^{\mu\nu}(\delta g^{\rho\sigma}\delta g_{\rho\sigma}-\frac{1}{2}(\delta g_\rho^\rho)^2)\right)\partial_5\delta g_{\mu\nu}\right. \\
&\quad \left.+\left(\delta g^{\mu\rho}\delta g_{\rho\sigma}\delta g^{\nu\sigma}-\frac{1}{2}\delta g_\rho^\rho\delta g^{\mu\sigma}\delta g_\sigma^\nu-\frac{1}{4}\delta g^{\mu\nu}(\delta g^{\rho\sigma}\delta g_{\rho\sigma}-\frac{1}{2}(\delta g_\rho^\rho)^2)\right)\partial_5\bar{g}_{\mu\nu}\right. \\
&\quad \left.-\frac{1}{6}\left(\delta g^{\mu\nu}\delta g_{\mu\rho}\delta g_\nu^\rho-\frac{3}{4}\delta g^{\mu\nu}\delta g_{\mu\nu}\delta g_\rho^\rho+\frac{1}{8}(\delta g_\rho^\rho)^3\right)\bar{g}^{\zeta\tau}\partial_5\bar{g}_{\zeta\tau}\right]. \tag{132}
\end{aligned}$$

The cubic action $\delta^{(3)}\mathcal{S}$ is given by the sum of $\delta^{(3)}\mathcal{S}_{\text{EH}} + \delta^{(3)}\mathcal{S}_{\text{GH}}$ and of the cubic expansion of the boundary counterterms, multiplied by the gravitational prefactor $N_c^2/(2^6\pi^2)$.

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